COALITION ROBUSTNESS OF MULTIAGENT SYSTEMS

by

Nghia C. Tran

A thesis submitted to the faculty of

Brigham Young University

in partial fulfillment of the requirements for the degree of

Master of Science

Department of Computer Science

Brigham Young University

April 2009

BRIGHAM YOUNG UNIVERSITY

GRADUATE COMMITTEE APPROVAL

of a thesis submitted by

Nghia C. Tran

This thesis has been read by each member of the following graduate committee and by majority vote has been found to be satisfactory.

Date	Sean C. Warnick, Chair

Date	Christophe Giraud-Carrier

Date	Quinn O. Snell

BRIGHAM YOUNG UNIVERSITY

As chair of the candidate's graduate committee, I have read the thesis of Nghia C. Tran in its final form and have found that (1) its format, citations, and bibliographical style are consistent and acceptable and fulfill university and department style requirements; (2) its illustrative materials including figures, tables, and charts are in place; and (3) the final manuscript is satisfactory to the graduate committee and is ready for submission to the university library.

_____   _____

Date   Sean C. Warnick
                                          Chair, Graduate Committee

Accepted for the
Department   _____

                                          Kent E. Seamons
                                          Graduate Coordinator

Accepted for the
College   _____

                                          Thomas W. Sederberg
                                          Associate Dean, College of Physical and Mathematical
                                          Sciences

ABSTRACT

COALITION ROBUSTNESS OF MULTIAGENT SYSTEMS

Nghia C. Tran

Department of Computer Science

Master of Science

ACKNOWLEDGMENTS

Contents

Chapter 1

Introduction

1.1 Organization of the Thesis

Chapter 2

The Value of Cooperation Within a Profit-Maximizing Organization

This paper proposes a measure to quantify the value of cooperation experienced by a firm. Using a reverse merger simulation approach, the percentage of a firms profits due to cooperation can be precisely determined. This is accomplished by considering the profit maximizing dynamics of firms in the market as defining a value function of a coalition game. A simple example illustrates the ideas.

2.1   Introduction

This paper derives a method for quantifying the value of cooperation (VC) and the relative value of cooperation (RVC) experienced by a firm. The idea is to let the profit maximizing dynamics of a given market structure define the value function for a particular coalition game. With this idea, we may aid anyone who needs to know about a products place in the product network. For business managers, this means they can know the products their business offers which contribute to a greater whole, as opposed to those product lines which may be sold off with minimal impact. They may also discover which product lines would be most advantageous for their business. Given any set of products inside or outside the business, we may calculate the value of this set (with profit maximization the objective).

Our method is also useful for the antitrust division of the Justice Department. They are interested maximizing total social welfare in a market by protecting market competition. To do this, they attempt to measure the control a particular company has on the market and

take appropriate measures. Their preferred measure of the market power of the company is the Herfindahl-Hirshman index (HHI) [7], the sum of the squares of each firms market share, given by

$$I_{HH} = \sum_{i=1}^{N} s_i^2 \tag{2.1}$$

This measure, however, relies on legal definitions of particular markets and focuses on a computation of market share. Market share, however, has been shown to be a weak indicator of market power [3]. A more direct measure of market power that is insensitive to legal definitions of market boundaries but highly sensitive to the economics of the underlying product network would make a significant impact on antitrust efforts. The value of cooperation a firm is able to realize within a given economic environment is a step in the direction of computing market power directly. This work draws heavily from the theory of industrial organization and coalition games[7], [1], [5], [6], [4]. The most closely related work to our study is recent work on merger simulations. One paper [2] describes how the impact of a proposed merger can be computed by evaluating the post-merger equilibrium prices. The paper considers common functional forms of demand functions, and indicates how to conduct the merger simulation in each case. The value of cooperation proposed here is found through a kind of reverse merger simulation that explores the impact of splitting the firm into its constituent economic units to determine the value it is realizing by unifying the objectives of these basic units.

The next section introduces the dynamic framework motivating the profit gained at equilibrium as a viable value function. A coalition game is then formulated using this value function, and the Value of Cooperation and Relative Value of Cooperation are then introduced as measures on this game. A simple example is then provided to illustrate the ideas, and the conclusion and future work summarizing the work follows.

4

2.2 Profit maximizing dynamics

Consider a market, $\mathscr{M}$, of N products. Without loss of generality, give these products an arbitrary order and integer label so that $\mathscr{M} = \{1, 2, ..., n\}$. Let $p \in \mathbb{R}^N$ be the vector of (non-negative) prices for these N products, and let $q : \mathbb{R}^N \to \mathbb{R}^N$ be the (non-negative) demand for these products at prices p.

A firm, F is a subset of the N products in the market, $F \in 2^{\mathscr{M}}$. This implies that the firm controls the production and distribution of the products assigned to it. Most importantly for our analysis, since we consider a Bertrand market model, this implies that the firm may set the prices of the $n = |F|$ products assigned to it.

We suppose that the products of the market are partitioned between m firms. This implies that no two firms control the same product, $F_i \cap F_j = \emptyset \;\; \forall i \neq j$ and that the union of all products assigned to the m firms compose the entire market, $\bigcup_{i=1}^{m} F_i = \mathscr{M}$.

Let $c_j(q_j), j = 1, ..., N$ be the cost of production of q_j units of product j. The profit of the ith firm is then given by

$$\pi_j = \sum_{j \in F_i} [q_j(p)p_j - c_j(q_j(p))]$$

A profit-maximizing firm under the Bertrand model of market behavior will tend to change its prices to maximize its short-term profit. We model this behavior by assuming that the firm will evolve the prices of its products in the direction of maximally improving its profits. That is, if product j belongs to firm i, then we expect the firm to evolve the price of product j as

$$\frac{dp_j(t)}{dt} = \left. \frac{\partial \pi_i(p)}{\partial p_j} \right|_{p(t)}$$

where $p(t)$ is the pricing vector for the entire market at time t.

Notice that these dynamics suggest that if the partial derivative of profits is negative with respect to the price of product j, that the firm should *decrease* the price of product j.

This is in the direction of improving profits. Likewise, if the partial derivative were positive, the firm would increase the price of product j to improve profits. When the partial derivative is zero, the motivation is to hold the price at this locally profit-maximizing position.

Reordering the N market products so that each firms products are grouped together, and letting n_i be the number of products controlled by firm i, we then can partition the pricing vector into components associated with each firm. If every firm in the market is assumed to be profit maximizing, this yields the following market dynamics:

$$
\begin{bmatrix}
\dot{p}_1(t) \\
\vdots \\
\dot{p}_{n_1}(t) \\
\hline
\dot{p}_{n_1+1}(t) \\
\vdots \\
\dot{p}_{n_1+n_2}(t) \\
\hline
\vdots \\
\dot{p}_{1+\sum_{i=1}^{m-1} n_i}(t) \\
\vdots \\
\dot{p}_N(t)
\end{bmatrix}
=
\begin{bmatrix}
(\partial \pi_1/\partial p_1)(p(t)) \\
\vdots \\
(\partial \pi_1/\partial p_{n_1})(p(t)) \\
\hline
(\partial \pi_2/\partial p_{n_1+1})(p(t)) \\
\vdots \\
(\partial \pi_2/\partial p_{n_1+n_2})(p(t)) \\
\hline
\vdots \\
(\partial \pi_m/\partial p_{1+\sum_{i=1}^{m-1} n_i})(p(t)) \\
\vdots \\
(\partial \pi_m/\partial p_N)(p(t))
\end{bmatrix}
\tag{2.2}
$$

where the dot notation $\dot{p}(t)$ is used to represent $dp(t)/dt$. Notice that if the market system (2) has an equilibrium, such a pricing vector peq would represent prices from which no firm can improve its profits by unilaterally changing the prices over which it has control. Under certain technical conditions such an equilibrium can be shown to exist. Moreover, this equilibrium can often be shown to be asymptotically stable, in the sense that any pricing vector $p(0)$ will converge to the equilibrium p_{eq} as $t \to \infty$.

2.3 The firm as a Coalition in a Multi-Coalition Environment

Under the assumption that the market dynamics are stabilizing, we expect price perturbations to re-equilibriate. In this context, it is convenient to simplify the problemby only considering the profits of the firms at equilibrium. These profits define a payoff function reminiscent of those used to define coalition games.

Let $v(Fi) = \pi_i|_{p=p_{eq}}$ be the payoff or profit of firm i at the market equilibrium prices peq. In this way the firm may be thought of as a coalition of n_i players in an N-player cooperative game. Each player is a one-product company that completely manages the production, distribution, and pricing decisions for its product. The firm, then, is a confederacy of these one-product companies that works together to maximize their combined profits or payoffs.

The theory of coalition games studies the behavior of such coalitions once the payoff function is defined for every possible coalition. The idea is that any given coalition F_i yields a well-defined payoff $v(F_i)$, and then a number of questions can be explored regarding how to distribute the payoff among the members of the coalition, etc.

Our situation is different because the payoff to a given firm doesnt just depend on the products it controls, but also on the market structure of the products outside the firm. For example, consider a 10-product market and a three product firm in the market. The payoff to the firm does not just depend on the prices of the three products it controls, but also on the prices of the other seven products. The profit-maximizing equilibrium prices of these other seven products, however, may be set differently depending on whether they belong to a single firm or whether they are controlled by seven different companies. Thus, the payoff to the three-product firm depends on the entire market structure.

Coalition game theory addresses such situations by considering partition systems and restricted games. For our purposes, it is sufficient to partition the N products of M into m firms and then assume that this structure is fixed outside of the particular firm that we are studying. This enables us to work with a well defined payoff function induced by the profit-

maximizing dynamics of firms within the market without eliminating the multiple-coalition (i.e. multiple firm) cases of interest.

2.4 Value of Cooperation

To quantify the value of organizing a group of one-product companies into a single firm, we need to compare the profits the firm receives if it sets its prices as if each of its products were independent companies with those it realizes by fully capitalizing on cooperation between the products. More precisely, let p_{eq} be the profit-maximizing equilibrium prices for the given market structure. In contrast, consider the new profit maximizing equilibrium prices achieved without cooperation if F_i were divided into its constituent one-product companies and each independently optimized their prices. Let this second set of equilibrium prices serve as a basis for comparison, or reference, and be denoted p_{ref}. The value of cooperation (VC) of a given firm F_i in market $\mathcal{M}$ with structure $\mathcal{S} = \{F_1, F_2, ..., F_m\}$ is then given by

$$\text{VC}_{\text{ref}}(F_i, S) = \pi_i|_{p_{eq}} - \pi_i|_{p_{ref}} \tag{2.3}$$

This measure captures precisely the value realized by the firm due to cooperation within its organization. Nevertheless, the measure carries units of dollars and reflects a kind of absolute dollar-value of cooperation within the firm. Moreover, note that the measure is always non-negative since the cooperating firm can always recover at least the non-cooperating, or reference, profits by simply setting the prices it controls in peq to those of pref .

A related measure captures the relative value of cooperation (RVC) by normalizing VC_{ref} by the equilibrium profits as:

$$\text{RVC}_{\text{ref}}(F_i, S) = \frac{\pi_i|_{p_{eq}} - \pi_i|_{p_{ref}}}{\pi_i|_{p_{eq}}} \tag{2.4}$$

This measure is interpreted as the percentage of profits due to cooperation within the organization. It is bounded between zero and one, and it facilitates direct comparison between firms of different sizes.

Sometimes we may be interested in measuring the value of cooperation between structures other than the current market structure and the reference structure. This could be the case when considering mergers between firms, or when management is considering selling off a piece of the firm. In such cases it is easy to extend the definitions of VC and RVC by simply replacing the equilibrium and reference prices with the equilibriated profit-maximizing prices of the two market structures being compared.

It is instructive to contrast the VC and RVC with other measures used to characterize cooperative games. Hart and Mas-Colell defined a measure, called the potential, P, that computes the expected normalized worth of the game i.e. the per-capita potential, P/N, equals the average per-capita worth $(1/m)\sum_i(\pi_i)/(|Fi|)$. Given a market structure, this measure characterizes the expected profit of an averagesized firm (where size is measured with respect to the number of products the firm controls) in the market, even if such a firm does not actually exist.

Moreover, the potential has been connected to another measure, called the Shapley value, Φ_j , which yields the marginal contribution of each product in the market. This measure characterizes how the payoff of a coalition should be divided between members of the team. In both cases, the potential and Shapley value do not suggest anything about the intrinsic benefit of forming coalitions in the first place.

The Value of Cooperation, VC, and Relative Value of Cooperation, RVC, on the other hand, capture the natural significance for organizing production into multi-product firms. Nevertheless, these measures do not yield any information about how the profit of a firm should be efficiently invested into each of the firms constituent production lines. Thus, the measures are inherently different from the potential or shapely value of the coalition

game that focus more on the value of a member of a coalition to the group rather than the value of the coalition as a whole.

2.5 Example

To illustrate the point, consider a two product economy with linear demand given by

$$
\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \begin{bmatrix} -3.5 & -1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} + \begin{bmatrix} 100 \\ 100 \end{bmatrix} \tag{2.5}
$$

Suppose that the unit production cost of each product is $c_1 = 10, c_2 = 10$. If we consider a market structure where each product is produced by an independent company, the profit function for each company becomes

$$
\begin{aligned}
\pi_1(t) &= q_1(t)(p_1(t) - c_1) \\
&= -3.5p_1^2 - p_1 p_2 + 135p_1 + 10p_2 - 1000 \tag{2.6} \\
\pi_2(t) &= q_2(t)(p_2(t) - c_2) \\
&= -2p_2^2 - 3p_1 p_2 + 30p_1 + 120p_2 - 1000 \tag{2.7}
\end{aligned}
$$

Taking the partial derivatives of each profit function with respect to the appropriate pricing variable, we find the profit-maximizing market dynamics to be:

$$
\begin{bmatrix} \frac{dp_1(t)}{dt} \\ \frac{dp_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} -7 & -1 \\ -3 & -4 \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} + \begin{bmatrix} 135 \\ 120 \end{bmatrix} \tag{2.8}
$$

Figure 1 shows how the two-firm dynamics drive an initial pricing vector to a profit-maximizing equilibrium. This equilibrium price is

$$p_{ref} = \begin{bmatrix} 16.8 \\ 17.4 \end{bmatrix}$$

and the associated equilibrated profits are $\pi_1 = 161.84$, and $\pi_2 = 109.52$.

Now, consider a market structure where both products are controlled by the a same firm. In this case, the firms profit function becomes

$$\pi(t) = q_1(t)(p_1(t) - c_1) + q_2(t)(p_2(t) - c_2)$$
$$= -3.5p_1^2 + 165p_1 - 4p_1p_2 + 130p_2 - 2p_2^2 - 2000 \qquad (2.9)$$

With this market structure, the firm adjusts the prices of both products to optimize the same objective. These new dynamics become:

$$\begin{bmatrix} \frac{dp_1(t)}{dt} \\ \frac{dp_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} -7 & -4 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} + \begin{bmatrix} 165 \\ 130 \end{bmatrix} . \qquad (2.10)$$

Figure 2 shows how the single-firm dynamics drive an initial pricing vector to a profit-maximizing equilibrium. The new equilibrium price is given by

$$p_{eq} = \begin{bmatrix} 11.67 \\ 20.83 \end{bmatrix} \qquad (2.11)$$

11

and the associated equilibriated profits are $\pi_{eq} = 316.667$. The value of cooperation in this example thus becomes

$$\text{VC} = \pi_{eq} - (\pi_1 + \pi_2) = 45.3067$$

$$\text{RVC} = 0.1431$$

This suggests that in this market, just under 15% of the profits of the two-product firm are the direct result of its interfirm cooperation.

2.6 Conclusion

This paper explored quantitative measures to calibrate the value of cooperation within a specific firm in a given market. The idea is to assume profit-maximizing dynamics among the firms within the market and compare equilibrium profits in two different scenarios. The first scenario considers the firm as it is, as a single economic entity with a unified objective and exhibiting full cooperation between its various economic units. The second scenario considers splitting the firm into its constituent economic units and computing market equilibrium prices if these units were to fail to cooperate and acted completely independently out of self interest. The difference between the cooperative profits of the first scenario and the aggregate profits of the independent units of the second scenario define a measure we call the Value of Cooperation, VC, of the firm in its current market environment. A second related measure is the Relative Value of Cooperation, RVC, which normalizes the VC measure by the cooperative profits to yield a unitless metric that reveals the percentage of profits derived from cooperation within the firm.

Quantifying the value of cooperation is a first step in understanding how firms exert market power in their respective environments. This information is important for both managers, who hope to leverage the information to better lead their organizations, and regulators, who want to monitor the impact of corporate decisions on social welfare. Future

work will concretely establish the relationship between the VC and RVC measures and market power and indicate how to compute approximations to these metrics from readily available market data.

Chapter 3

Cooperation-based Clustering for Profit-maximizing Organizational Design

This paper shows how the notion of value of cooperation, a measure of the percentage of a firm's profits due strictly to the cooperative effects among the goods it sells, can be used to analyze the relative economic advantage afforded by various organizational structures. The value of cooperation is computed from transactions data by solving a regression problem to fit the parameters of the consumer demand function, and then simulating the resulting profit-maximizing dynamic system under various organizational structures. A hierarchical agglomerative clustering algorithm can be applied to reveal the optimal organizational substructure.

3.1 Introduction

Analyzing the impact of organizational structure on the performance of profit-maximizing organizations is a difficult task for business managers. Yet, informed design decisions are essential to long-term profitability. While these decisions are often ad hoc, today's large volumes of data make more systematic analyses possible. The key concept in such analyses is that of the value of cooperation (VC) experienced by a firm, which captures the percentage of a firm's profits due strictly to the cooperative effects among the goods it sells [1].

Such a measure provides the scientific backing for sound organizational design decisions. For example, if a firm can identify which of its products have strong synergies with others, it can organize to ensure that decision makers of related products work closely together. This may include physically co-locating entities where interaction adds strong value to the organization, or it may result in decentralization when cooperation adds little value. Similarly, if a firm identifies pieces of its business that add little cooperative benefit to the organization as a whole, it may consider selling off these subunits. A subunit with a healthy balance sheet may sell for a high price without adversely affecting the firm's value of cooperation. On the other hand, the firm may pursue a different strategy of retaining its decoupled subunits but use the value of cooperation to identify an acquisition that strongly couples their mutual benefit. Thus, divisions of a firm that are quite independent may be cooperatively coupled through the acquisition of another business with the right cooperative effects. For example, a firm with two distinct independent divisions that have no value of cooperation may acquire another business unit that not only adds value of cooperation with each division, but does so in a way that the total business becomes tightly integrated. Moreover, the firm may identify an acquisition candidate that is struggling on its own, and thus is inexpensive, but brings the right cooperative effects to the organization to offset the risk of acquiring a struggling business. The value of cooperation thus becomes the lens through which a firm can better identify valuable opportunities in the market environment, or costly "baggage" in its own organizational structure.

In this paper, we discuss a value of cooperation inspired by the theory of industrial organization and coalition games [2], [3], [4], [5], [6]. Essentially, the profit maximizing dynamics of a given organizational structure define the value function for a particular coalition game. Researchers in [7] describe how the impact of a proposed merger can be computed by evaluating the post-merger equilibrium prices. They consider common functional forms of demand functions and indicate how to conduct the merger simulation in each case. Our value of cooperation is computed through a kind of "reverse" merger sim-

16

ulation that explores the impact of splitting the firm into its constituent economic units to determine the value it is realizing by unifying the objectives of these basic units. We are then able to use this measure to drive a hierarchical agglomerative clustering algorithm in order to reveal the organizational substructure defined naturally from the cooperative effects of the global product network.

3.2   The Value of Cooperation

Consider a market, M, selling N products. A firm, F, is a subset of the N products in the market, $F \in 2^M$. This implies that the firm controls the production and distribution of the products assigned to it. Furthermore, we consider a Bertrand market model, so that each firm may set the prices of the products assigned to it. We assume that the products of the market are partitioned between m firms (i.e., no two firms control the same product) and that every firm in the market is a profit maximizing entity. Under the standard assumption that market dynamics are stabilizing, we expect price perturbations to re-equilibriate, which means that we may simply consider the profits of firms at equilibrium. These profits, in turn, define a payoff function reminiscent of those used to define coalition games.

Let $v(Fi) = \pi_i|_{p=p_{eq}}$ be the payoff or profit of firm i at the market equilibrium prices peq. In this way the firm may be thought of as a coalition of ni players in an N-player cooperative game. Each player is a one-product company that completely manages the production, distribution, and pricing decisions for its product. The firm, then, is a confederacy of these one-product companies that work together to maximize their combined profits or payoffs. The theory of coalition games studies the behavior of such coalitions once the payoff function is defined for every possible coalition. The idea is that any given coalition F_i yields a well-defined payoff $v(F_i)$, and then a number of questions can be explored regarding how to distribute the payoff among the members of the coalition, etc. Our situation is different, however, because the payoff to a given firm does not depend only on the products it controls, but also on the market structure of the products outside the firm.

For example, consider a 10-product market and a three product firm in the market. The payoff to the firm does not depend only on the prices of the three products it controls, but also on the prices of the other seven products. The profit-maximizing equilibrium prices of these other seven products, however, may be set differently depending on whether they belong to a single firm or whether they are controlled by seven different firms. Thus, the payoff to the three-product firm depends on the entire market structure. Coalition game theory addresses such situations by considering partition systems and restricted games. For our purposes, it is sufficient to partition the N products of M into m firms and assume that this structure is fixed outside of the particular firm that we are studying. This enables us to work with a well-defined payoff function induced by the profit-maximizing dynamics of firms within the market without eliminating the multiple-coalition (i.e., multiple firm) cases of interest.

To quantify the value of organizing a group of one-product companies into a single firm, we need to compare the profits the firm receives if it sets its prices as if each of its products were independent companies with those it realizes by fully capitalizing on cooperation between the products.

More precisely, let p_{eq} be the profit-maximizing equilibrium prices for the given market structure. In contrast, consider the new profit maximizing equilibrium prices achieved without cooperation if F_i were divided into its constituent oneproduct companies and each independently optimized their prices. Let this second set of equilibrium prices serve as a basis for comparison, or reference, and be denoted pref. We can then define the following measure.

Definition 1. *The Value of Cooperation (VC) of a firm F_i in market M with structure $S = F_1, F_2, ..., F_m$ is given by:*

$$\text{VC}_r ef(F_i, S) = \pi_i|_{p_{eq}} - \pi_i| p_{ref} \tag{3.1}$$

18

This VC measure captures precisely the value realized by the firm due to cooperation within its organization. Note that the VC measure is always non-negative since the cooperating firm can always recover at least the non-cooperating, or reference, profits by simply setting the prices it controls in peq to those of p_{ref}.

As defined, the VC measure carries units of dollars and reflects a kind of absolute dollar-value of cooperation within the firm, thus making comparisons difficult. We, therefore, define a "relative" Value of Cooperation by normalizing VC$_r ef$ by the equilibrium profits as follows.

Definition 2. *The Relative Value of Cooperation (RVC) of a firm F_i in market M with structure $S = F_1, F_2, ..., F_m$ is given by:*

$$\text{RVC}_{ref}(F_i, S) = \frac{\pi_i\big|_{p_{eq}} - \pi_i\big|_{p_{ref}}}{\pi_i\big|_{p_{eq}}} \tag{3.2}$$

This RVC measure is naturally interpreted as the percentage of profits due to cooperation within the organization. It is bounded between zero and one, and enables direct comparison among firms of different sizes. By simply replacing the equilibrium and reference prices in the above definitions with the equilibriated profitmaximizing prices of the market structures being compared, one can easily use the thus modified VC and RVC to analyze the relative values of different organizational structures within a single firm. This is a natural application of the above framework, where the market is a firm and the firms are its organizational divisions.

Chapter 4

Coalition Robustness of Multiagent Systems

Business networks provide one of the most compelling environments to study the conflicting effects of competition and cooperation on multi-agent dynamical systems. While firms engage various merger and divestiture strategies to create the desired cooperative environment that enhances their market power, governmental regulatory agencies enforce antitrust measures that protect competition as a means to limit the market power of these organizations. Merger simulation has subsequently evolved in recent years as a mechanism to study the impact of different organizational structures on the market. Nevertheless, typical economic models can often lead to competition dynamics that arbitrarily lose stability when considering different organizational structures. This work provides stability robustness conditions with respect to coalition structure for profit-maximizing dynamical systems with network demand, and partially convex utility. In particular, we show that stability of the coalition of all agents is sufficient to guarantee stability of all other coalition structures. These conditions are then leveraged to provide a systematic methodology for estimating a rich variety of demand systems that guarantee sensible stability results regardless of the structure of cooperation in the marketplace.

4.1 Background: Firms, Market Power, and Merger Simulation

One of the most well-studied multi-agent systems is the marketplace. Market dynamics are governed by competition, nevertheless one of the most interesting features of the market is

the spontaneous emergence of cooperation structures we call firms. Firms represent coalitions of agents that offset the computational limitations of individual agents to better compete for scarce resources. They orchestrate policies that attempt to drive profit-generating dynamics in the face of considerable uncertainty, both from the consumer market and from the competitive forces of other firms. Piloting a firm is one of the most interesting and difficult control problems we've encountered.

One way firms cope with market uncertainty is through growth. As firms deploy successful policies, they acquire capital that enable them to attract the cooperation of more agents in the marketplace. This can happen organically through the hiring of employees and the natural expansion of the firm's existing operations, or it can happen suddenly through mergers and acquisitions. Either way, such growth attempts to mitigate uncertainty by either entrenching the firm in the market niche known to have been previously successful, or by offsetting risk by diversifying the types of products or services the firm uses to compete for profits.

As firms generate wealth, they distribute a portion of it to their stake holders, who then engage the marketplace as consumers or investors of one kind or another. The ability of consumers to translate this wealth into an improved quality of life, however, depends significantly on the balance of power between firms in the marketplace. When firms are too strong, they do not have the incentive to innovate, and they can restrict the flow of existing goods and services to consumers unless premium prices are paid. When firms are too weak, they do not have the ability to innovate, nor do they generate the wealth their stake holders might otherwise have had to participate more fully as consumers or further investors in the marketplace. As a result, governments control the growth and strength of firms, either by stopping proposed mergers or by forcing firms to divide. This maintains competition as an effective force to limit the market power of firms, and it ideally creates resonance between the welfare of consumers and the welfare of investors that fuel growth.

At the heart of both the firm's growth strategy and the government's regulation strategy, then, lies the ability to measure a firm's market power. In 1997 the US Department of Justice and the Federal Trade Commission's released guidelines governing the regulation of mergers within the United States [1]. This, in turn, precipitated growing interest in the use of "merger simulations" to estimate the effects of proposed mergers or acquisitions [6], [7], [13], [2] and [3].

Merger simulations predict post-merger prices based on a demand model of the relationship between prices charged and quantities sold by the firms under investigation in the relevant market. Assumptions or models about supply issues are also incorporated into the simulation. Under a Bertrand model of pricing, every firm sets the prices of its brands to maximize its profits. Equilibrium results when no firm can unilaterally change its prices to improve its profits. Simulations compare pre-merger prices and profits with post-merger prices and profits to analyze the impact of the merger. "Reverse" simulations compare prices and profits of an existing firm with those resulting from the division of the firm into constitutive components, thereby measuring the "Value of Cooperation" achieved by the strategic positioning of the firm as the coalition of those particular components within the context of the larger market [9], [8], and [10].

In this way, Value of Cooperation can be viewed as a quantification of market power, and merger simulation can be thought of as a Value of Cooperation measurement on the post-merger firm. The presence of market power alone, however, is not necessarily illegal, nor is it sufficient to give the firm monopolistic power, as the firm would also need to create barriers of entry to prevent new firms from competing. Likewise, there may be other measures used to quantify the impact of market structure or industrial organization on market conditions. Nevertheless, such measures typically compare a property of an equilibrium of one market structure with that resulting from a different market structure, and are thus comparative static analyses that typically ignore dynamic issues.

Often, however, the demand models used in such simulations can lead to unstable equilibria, or even conditions where no equilibria exist at all for some market structures [3]. Such results are generally not the foreshadows of pending market doom should the right conspiracy be formed, but rather are simply dynamic limitations resulting from mathematical technicalities of the these models. None of the demand models typically used in economics, i.e. linear, log-linear (constant elasticity), logit, AIDS, and PCAIDS, guarantee the existence and stability of equilibria for all possible market structures.

Viewing the marketplace as a profit-maximizing multi-agent dynamical system (Section II), this work resolves these issues by providing stability robustness conditions with respect to coalition structure for such systems when these systems have a particular network demand structure (Section III). These conditions are then leveraged to provide a systematic methodology for empirically estimating a rich variety of AIDS-like demand systems from market data, using standard convex-optimization tools, that guarantee sensible stability results regardless of the structure of cooperation in the marketplace (Section IV).

4.2   Markets as Multi-Agent Systems

Consider a market consisting of n products, each produced and controlled by a single product division. These product divisions are the constitutive agents in our multi-agent system, N, and they are arbitrarily ordered and numbered 1 to n. Following a Bertrand model of pricing, each agent has complete authority and control to price its product as it sees fit. The prices for all the products are public knowledge, known at any given time by all the agents, and denoted by the vector $x \in \mathbb{R}^n$. For convenience, we will assume that the prices are in units relative to the unit cost of production for each product. That is, x_i is the markup for product i.

We suppose that the aggregate effect of consumers in the market is given by a demand function, $q(x) : \mathbb{R}^n \to \mathbb{R}^n$, which characterizes how the quantity sold for each product

varies with prices. Note that the demand, $q_i(x) : \mathbb{R}^n \to \mathbb{R}$, for product i depends, in general, not only on its own price, but on the prices of all the other products as well.

Each agent is equipped with a utility function that scores its reward as a function of the decisions of all the agents in the system. This utility function is a component of the market utility and is given by each product division's profits:

$$U_i(x) = x_i q_i(x). \tag{4.1}$$

A firm, F, is a coalition of agents, represented as a subset of N. We allow the market to coalesce into $m \leq n$ firms, where every agent belongs to one and only one firm. Thus, the market structure, or industrial organization, $\mathscr{F} = \{F_1, F_2, ..., F_m\}$, is a partition of N. We will write $\mathscr{F}^{-1}(i)$ for the firm to which agent i belongs.

We associate with each firm an objective or profit function given by the sum of the utility functions of the agents belonging to the firm,

$$U_F(x) = \sum_{i \in F} U_i(x) = \sum_{i \in F} x_i q_i(x). \tag{4.2}$$

By associating with a firm, an agent agrees to adjust the prices of its product to maximize the total profits or objective of the firm, rather than simply maximize its own utility. Thus, all agents belonging to the same firm adopt a common objective and effectively surrender their pricing authority to the firm, allowing the firm to lose money by underpricing in one division in order to induce a greater demand and profit in another division.

Each agent therefore changes its price in the direction of the gradient of the objective of the firm to which it belongs;

$$\dot{x}_i = \frac{\partial U_F}{\partial x_i}(x) = \frac{\partial \left[\sum_{i \in F} U_i\right]}{\partial x_i} = \sum_{i \in F} \frac{\partial U_i}{\partial x_i}(x). \tag{4.3}$$

Substituting from (4.1) for the profit structure of an agent's utility and writing them in vector notation, these dynamics become

$$\dot{x} = V_{\mathscr{F}}(x) = \left[D_{\mathscr{F}} \left(J_q^T(x) \right) \right] x + q(x), \tag{4.4}$$

where $J_q(x)$ is the Jacobian of the function $q(x)$, A^T denotes transpose of a matrix A, and $D_{\mathscr{F}}(A)$ is defined as: a) $d_{ij} = a_{ij}$ if $j \in \mathscr{F}^{-1}(i)$, and b) $d_{ij} = 0$ otherwise. Thus, if $\mathscr{F} = \{(1,2),3\}$ and A were given by

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad \text{then} \quad D_{\mathscr{F}}(A) = \begin{bmatrix} 1 & 2 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix}.$$

Given a market structure and a demand function, Equation (4.4) thus represents the profit-maximizing dynamics of the multi-agent system and becomes the central focus of our analysis.

Our stability robustness problem, then, is to find conditions under which we can guarantee existence, uniqueness and stability of the equilibrium of Equation (4.4) for all market structures $\mathscr{F} \in \Delta$, where Δ is the set of all partitions of N.

Example 1. *Consider a market with three products with consumer demand given by:*

$$\begin{bmatrix} q_1(x) \\ q_2(x) \\ q_3(x) \end{bmatrix} = \begin{bmatrix} -3 & -5 & 4 \\ -4 & -4 & 3 \\ 1 & 2 & -15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 80 \\ 90 \\ 80 \end{bmatrix} \tag{4.5}$$

Note that the demand is linear, and based on the signs of coefficients in the demand function, we can see that Products (1 and 2) are compliments, while (1 and 3) and (2 and 3) are substitutes. That is to say, an increase in the price of Product 1 results in decreased sales

of both Products 1 (as you would expect) and 2 (i.e it is a compliment to Product 1), but an increase of sales of Product 3 (i.e. it is a substitute for Product 1).

The utility functions of the constitutive agents, meaning the three product divisions that each control a single product, are thus given by

$$
\begin{aligned}
U_1 &= (-3x_1 - 5x_2 + 4x_3 + 80)x_1 \\
U_2 &= (-4x_1 - 4x_2 + 3x_3 + 90)x_2 \quad . \\
U_3 &= (x_1 + 2x_2 - 15x_3 + 80)x_3
\end{aligned}
\tag{4.6}
$$

Moreover, given any market structure $\mathscr{F}$, the profit-maximizing dynamics of this multi-agent system then become

$$
\dot{x} = D_{\mathscr{F}} \left(\begin{bmatrix} -3 & -4 & 1 \\ -5 & -4 & 2 \\ 4 & 3 & -15 \end{bmatrix} \right) x + q(x).
\tag{4.7}
$$

Now, lets compare the market dynamics for two different industrial organizations. First, we will consider the organization where every product division is its own firm, $\mathscr{F} = \{1,2,3\}$. In this case, the dynamics become:

$$
\dot{x} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -15 \end{bmatrix} x + q(x)
$$

$$
\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -6 & -5 & 4 \\ -4 & -8 & 3 \\ 1 & 2 & -30 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 80 \\ 90 \\ 80 \end{bmatrix}
\tag{4.8}
$$

It is easy to verify that this system has a stable equilibrium at $x = (8.91, 8.11, 3.50)$ dollars.

The demand at this point becomes $q = (26.72, 32.42, 52.63)$ units sold per unit time, and

the profits for each firm are $U = (238.07, 262.93, 184.21)$ dollars per unit time.

Now let's consider the organization where Divisions 1 and 2 merge to form a single

firm. This market structure is given by $\mathscr{F} = \{(1,2),3\}$, and the corresponding dynamics

become:

$$\dot{x} = \begin{bmatrix} -3 & -4 & 0 \\ -4 & -4 & 0 \\ 0 & 0 & -15 \end{bmatrix} x + q(x)$$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -6 & -9 & 4 \\ -8 & -8 & 3 \\ 1 & 2 & -30 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 80 \\ 90 \\ 80 \end{bmatrix} \quad (4.9)$$

We see that with these dynamics the system has an equilibrium at $x = (6.40, 6.08, 3.29)$

dollars, corresponding to the demand of $q = (43.56, 49.95, 49.21)$ units sold per unit time

and profits for the two firms of $U = (582.48, 161.90)$ dollars per unit time. Nevertheless,

this equilibrium point is unstable. As a result, these equilibrium values are never really

attainable, the profits of $582.48 for the merged firm can not actually be realized, because

even small changes in prices will lead, according to this model, to a never ending price war

that never converges. Note that there is no way to detect a priori that this particular market

structure would be unstable with this particular demand system. The merger of Divisions

1 and 3, for example, corresponding to market structure $\mathscr{F} = \{(1,3),2\}$, is stable.

■

4.3 Stability Robustness Conditions

Example 1 demonstrates how otherwise reasonable models of market dynamics can fail

when considering industrial organization issues. The demand model, which is of sufficient

fidelity to address questions such as the complimentary/substitutive relationship between products, drives the prediction that one possible merger will result in prices going to infinity. In reality, such a merger would not result in continually increasing prices; this result is simply an artifact of the model we have chosen. As a result, we see that this model is simply inadequate to describe market dynamics under changes in market structure, at least for some structures.

Nevertheless, if a model breaks down for some market structures by predicting unstable equilibria (or the lack of any equilibria, as happens for constant-elasticity models), can it be trusted to yield accurate results for any market structure? Whatever simplifications in the model cause it to drastically fail for some market structures might degrade its representation of the true dynamics under other market structures. The only safe course is to identify models that have sufficient fidelity to yield sensible results for every possible market structure.

Note that verifying the fidelity of a proposed model by checking the stability properties for all possible market structures is intractable; the number of possible market structures grows worse than exponential with n, the number of products, and real markets can involve thousands of products. As a result, we need tractable robustness conditions that can guarantee existence, uniqueness and stability of equilibria regardless of market structure.

To generate such conditions, we begin by defining the quantities we will use to check stability robustness of the system (4.4). For notational convenience let $F(i) = \mathscr{F}^{-1}(i)$ denote the firm to which the ith agent belongs. When introducing the lemmas, we will write $M_{m,n}(\mathbb{F})$ for the set of all $m \times n$ matrices whose entries are elements of the field $\mathbb{F}$, and we will abbreviate to $M_n(\mathbb{F})$ in the case of square matrices. For any square matrix $A \in M_n(\mathbb{C})$, we will denote its numerical range as $W(A) = \{x^*Ax \mid \|x\|_2 = 1\}$, and its spectrum as $\sigma(A)$. For a subset S of a vector space, we will write $\text{co}(S)$ to denote its convex hull. For two subsets A and B of a group $(G, +)$, we write $A + B = \{x + y \mid x \in A, y \in B\}$.

29

Lemma 1. *Given the system (4.4), the Jacobian of the system dynamics, $V_{\mathscr{F}}$, decomposes as:*

$$J_{V_{\mathscr{F}}}(x) = \left[A(x) + D_{\mathscr{F}}(A^T(x))\right] + B_{\mathscr{F}}(x) + C_{\mathscr{F}}(x), \tag{4.10}$$

where $A(x)$, $B_{\mathscr{F}}(x)$, and $C_{\mathscr{F}}(x)$ are given as follows:

$$A(x): \quad A_{ii}(x) = \tfrac{1}{2}\sum_{j=1}^{n}\frac{\partial^2 U_j}{\partial x_i^2}(x), \quad A_{ij}(x) = \frac{\partial^2 U_i}{\partial x_i \partial x_j}(x) \tag{4.11}$$

$$B_{\mathscr{F}}(x): \quad B_{ii}(x) = 0, \quad B_{ij}(x) = \sum_{k \in F(i)\setminus\{i,j\}} \frac{\partial^2 U_k}{\partial x_i \partial x_j}(x) \tag{4.12}$$

$$C_{\mathscr{F}}(x): \quad C_{ii}(x) = -\sum_{j \notin F(i)} \frac{\partial^2 U_j}{\partial x_i^2} \quad C_{ij}(x) = 0 \tag{4.13}$$

Proof. The diagonal entries of $J_{V_{\mathscr{F}}}(x)$ are given by,

$$
\begin{aligned}
J_{ii}(x) = \frac{\partial V_i}{\partial x_i}(x) &= \sum_{j \in F(i)} \frac{\partial^2 U_j}{\partial x_i^2}(x) \\
&= \sum_{j=1}^{n} \frac{\partial^2 U_j}{\partial x_i^2}(x) - \sum_{j \notin F(i)} \frac{\partial^2 U_j}{\partial x_i^2}(x) \\
&= 2A_{ii}(x) + C_{ii}(x) = 2A_{ii}(x) + B_{ii}(x) + C_{ii}(x).
\end{aligned} \tag{4.14}
$$

For $j \neq i$, the off-diagonal $J_{ij}(x)$ is given by,

$$
\begin{aligned}
J_{ij}(x) = \frac{\partial V_i}{\partial x_j}(x) &= \sum_{k \in F(i)} \frac{\partial^2 U_k}{\partial x_i \partial x_j}(x) \\
&= \sum_{k \in F(i)\cap\{i,j\}} \frac{\partial^2 U_k}{\partial x_i \partial x_j}(x) + \sum_{k \in F(i)\setminus\{i,j\}} \frac{\partial^2 U_k}{\partial x_i \partial x_j}(x) \\
&= \sum_{k \in F(i)\cap\{i,j\}} \frac{\partial^2 U_k}{\partial x_i \partial x_j}(x) + B_{ij}(x).
\end{aligned} \tag{4.15}
$$

When $j \in F(i)$, we then have

$$J_{ij}(x) = \frac{\partial^2 U_i}{\partial x_i \partial x_j}(x) + \frac{\partial^2 U_j}{\partial x_i \partial x_j}(x) + B_{ij}(x)$$

$$= A_{ij}(x) + A_{ji}(x) + B_{ij}(x)$$

$$= A_{ij}(x) + A_{ji}(x) + B_{ij}(x) + C_{ij}(x). \tag{4.16}$$

Otherwise, when $j \notin F(i)$, we then have

$$J_{ij}(x) = \frac{\partial^2 U_i}{\partial x_i \partial x_j}(x) + B_{ij}(x)$$

$$= A_{ij}(x) + B_{ij}(x) + C_{ij}(x) \tag{4.17}$$

Therefore,

$$J_{V_{\mathscr{F}}}(x) = \left[A(x) + D_{\mathscr{F}}(A^T(x))\right] + B_{\mathscr{F}}(x) + C_{\mathscr{F}}(x). \tag{4.18}$$

$\square$

Definition 3. *The market structure consisting of a single firm, $\mathscr{F} = \{(1,2,...,n)\}$, that is, where all agents belong to the same coalition, is called the* Grand Structure, *denoted $\mathscr{G}$, and the associated firm is called the* Grand Coalition, *denoted G.*

Lemma 2. *Given by (4.13) and Definition 3, $C_{\mathscr{G}}(x) = 0$.*

Proof. This follows directly from the definition of $C_{\mathscr{F}}$ in (4.13), where the only nonzero elements are on the diagonal, and the diagonal elements become zero for the Grand Structure since all agents belong to the same firm. $\square$

31

Definition 4. *A function* $h(x) : \mathbb{R}^n \to \mathbb{R}^m$ *is said to have* network structure *if there exist functions* $f_{ij} : \mathbb{R}^2 \to \mathbb{R}$ *such that*

$$h_i(x) = \sum_{j=1}^n f_{ij}(x_i, x_j), \quad i = 1, \ldots, n. \tag{4.19}$$

Lemma 3. *A demand function,* $q(x) : \mathbb{R}^n \to \mathbb{R}^n$, *with network structure induces network structure on the market utility function, given by (4.1), and the objective function of each firm in the market, given by (4.2).*

Lemma 4. *If the utility function,* $U(x)$, *associated with system (4.4) has network structure, then* $B_{\mathscr{F}}(x) = 0$ *for all market structures* $\mathscr{F}$.

Proof. Network structure of $U(x)$ implies there exist functions $f_{ij} : \mathbb{R}^2 \to \mathbb{R}$ such that $U_i(x) = \sum_{j=1}^n f_{ij}(x_i, x_j), \quad i = 1, \ldots, n$. Hence, for $k \notin \{i, j\}$,

$$\begin{aligned}
\frac{\partial^2 U_k}{\partial x_i \partial x_j}(x) &= \sum_{l=1}^n \frac{\partial^2 f_{kl}(x_k, x_l)}{\partial x_i \partial x_j} \\
&= \frac{\partial^2 f_{ki}(x_k, x_i)}{\partial x_i \partial x_j} + \frac{\partial^2 f_{kj}(x_k, x_j)}{\partial x_i \partial x_j} \\
&= \frac{\partial}{\partial x_i} \frac{\partial f_{ki}(x_k, x_i)}{\partial x_j} + \frac{\partial}{\partial x_j} \frac{\partial f_{kj}(x_k, x_j)}{\partial x_i} = 0.
\end{aligned}$$

Therefore, following from (4.12), $B_{\mathscr{F}}(x) = 0$ for all market structures $\mathscr{F}$. $\qquad\square$

Definition 5. *A utility function* $U(x) : \mathbb{R}^n \to \mathbb{R}^n$ *is said to be* partially convex *if,*

$$\frac{\partial^2 U_j}{\partial x_i^2}(x) \geq 0 \quad \forall j \neq i, \ \forall x \in \mathbb{R}^n. \tag{4.20}$$

Lemma 5. *When utility functions of the system (4.4) are partially convex,* $C_{\mathscr{F}}(x)$ *is a negative semidefinite diagonal matrix.*

Definition 6. *An n-product market with profit-maximizing dynamics given by (4.4), with demand function $q(x) : \mathbb{R}^n \to \mathbb{R}^n$ that has network structure, and with partially convex utility is said to be an* industrial organization network *for any market structure $\mathscr{F}$.*

Definitions 1 through 4 equip the models we will use to represent market dynamics with the technical structure we will need to guarantee stability robustness for all industrial organizations. In particular, industrial organization networks provide a model class with sufficient fidelity to explore questions involving changes in market structure. The following lemma comes from various parts in [4].

Lemma 6. *Let $A, B \in M_n(\mathbb{C})$.*

  (i) *$W(A)$ is compact and convex.*

 (ii) *$\text{co}(\sigma(A)) \subseteq W(A)$.*

(iii) *$W(A + B) \subseteq W(A) + W(B)$.*

(iv) *A is normal $\Rightarrow \text{co}(\sigma(A)) = W(A)$.*

Lemma 7. *For $A \in M_n(\mathbb{R})$,*

$$\max W(A + A^T) = \max \mathfrak{Re}W(A) + \max \mathfrak{Re}W(A^T).$$

Proof. Essentially follows from the definition of numerical range,

$$\max W(A + A^T) = \max_{\|x\|_2 = 1, x \in \mathbb{C}^n} x^*(A + A^T)x$$

$$= \max_{\|x\|_2 = 1, x \in \mathbb{C}^n} (x^*Ax + x^*A^Tx) = \max_{\|x\|_2 = 1, x \in \mathbb{C}^n} (x^*Ax + \overline{x^*Ax})$$

$$= 2 \max_{\|x\|_2 = 1, x \in \mathbb{C}^n} \mathfrak{Re}(x^*Ax) = 2 \max \mathfrak{Re}W(A).$$

Following the same reasoning, $\max W(A + A^T) = 2\max \mathfrak{Re}W(A^T)$, hence $\max W(A + A^T) = \max \mathfrak{Re}W(A) + \max \mathfrak{Re}W(A^T)$. $\square$

The following lemma is from [12].

Lemma 8. *Given $f : \mathbb{R}^n \to \mathbb{R}^n$, the equation $f(x) = y$ will have exactly one root for each y if there exist positive $\varepsilon, R \in \mathbb{R}$ such that for all $x \in \mathbb{R}^n$, $\|x\|_2 > R$,*

$$z^T \frac{\partial f}{\partial x}(x) z \leq -\varepsilon \|z\|_2^2 \quad \forall z \in \mathbb{R}^n.$$

Corollary 1. *Given $f : \mathbb{R}^n \to \mathbb{R}^n$, the equation $f(x) = y$ will have exactly one root for each y if there exists positive $\varepsilon \in \mathbb{R}$ such that,*

$$\max \mathfrak{Re} W \left(\frac{\partial f}{\partial x}(x) \right) \leq -\varepsilon \quad \forall x \in \mathbb{R}^n.$$

Proof. For all $z \in \mathbb{R}^n, \frac{z^T}{\|z\|_2} \frac{\partial f}{\partial x}(x) \frac{z}{\|z\|_2} \in W\left(\frac{\partial f}{\partial x}(x) \right)$, and also, $\frac{z^T}{\|z\|_2} \frac{\partial f}{\partial x}(x) \frac{z}{\|z\|_2} \in \mathbb{R}$, hence

$$\frac{z^T}{\|z\|_2} \frac{\partial f}{\partial x}(x) \frac{z}{\|z\|_2} \in W\left(\frac{\partial f}{\partial x}(x) \right) \cap \mathbb{R} \subseteq \mathfrak{Re} W\left(\frac{\partial f}{\partial x}(x) \right).$$

Thus, if $\max \mathfrak{Re} W \left(\frac{\partial f}{\partial x}(x) \right) \leq -\varepsilon$ then,

$$\frac{z^T}{\|z\|_2} \frac{\partial f}{\partial x}(x) \frac{z}{\|z\|_2} \leq \max \mathfrak{Re} W\left(\frac{\partial f}{\partial x}(x) \right) \leq -\varepsilon \Rightarrow z^T \frac{\partial f}{\partial x}(x) z \leq -\varepsilon \|z\|_2^2,$$

which satisfies the condition of Lemma 8. $\qquad\square$

Lemma 9. *For matrix $A \in M_n(\mathbb{R})$, $W\left(D_{\mathscr{F}}(A) \right) \subseteq W(A)$.*

Proof. For $F \subseteq \mathscr{F} = \{F_1, F_2, \ldots F_m\}$, let $I_F = \mathrm{diag}_{i=1}^n \left(\chi_F(i) \right)$, with $\chi_F(\cdot)$ being the membership function of F. Note that

$$D_{\mathscr{F}}(A) = \sum_{k=1}^m I_{F_k} A I_{F_k}.$$

Let $w \in W\big(D_{\mathscr{F}}(A)\big)$ and let $x \in \mathbb{C}^n$ such $\|x\|_2 = 1$ and $w = x^* D_{\mathscr{F}}(A)x$. Since $\sum_{k=1}^m I_{F_k} = I$, $\sum_{k=1}^m I_{F_k}x = x$, hence

$$1 = \|x\|_2^2 = x^*x = \left(\sum_{k=1}^m I_{F_k}x\right)^* \sum_{l=1}^m I_{F_l}x = \sum_{k=1,l=1}^m x^* I_{F_k} I_{F_l} x = \sum_{k=1}^m x^* I_{F_k}^2 x = \sum_{k=1}^m \|I_{F_k}x\|_2^2.$$

Let $\mathscr{F}^+ = \{F \in \mathscr{F} : I_F x \neq 0\}$. For $F \in \mathscr{F}^+$, let $y_F = \frac{I_F x}{\|I_F x\|_2}$. Therefore $\|y_F\|_2 = 1$ and $I_F x = \|I_F x\|_2 y_F$.

$$
\begin{aligned}
w = x^* D_{\mathscr{F}}(A)x = x^* \left(\sum_{F \in \mathscr{F}} I_F A I_F\right) x &= \sum_{F \in \mathscr{F}} x^* I_F A I_F x \\
&= \sum_{F \in \mathscr{F}} (I_F x)^* A (I_F x) = \sum_{F \in \mathscr{F}^+} (I_F x)^* A (I_F x) \\
&= \sum_{F \in \mathscr{F}^+} (\|I_F x\|_2 y_F)^* A (\|I_F x\|_2 y_F) \\
&= \sum_{F \in \mathscr{F}^+} \|I_F x\|_2^2 (y_F^* A y_F), \quad\quad\quad\quad (4.21)
\end{aligned}
$$

while $\sum_{F \in \mathscr{F}^+} \|I_F x\|_2^2 = \sum_{F \in \mathscr{F}} \|I_F x\|_2^2 = 1$. Therefore, w is a convex combination of $y_F^* A y_F$, which are in $W(A)$ because $\|y_F\|_2 = 1$. $W(A)$ is convex (Lemma 6) $\Rightarrow w \in W(A)$. $\qquad \square$

These lemmas demonstrate intermediate results that will enable us to provide stability robustness conditions for profit-maximizing dynamics under any coalition structure. In particular, Lemma 8 and Corollary 1 provide the machinery used to guarantee existence and uniqueness of an equilibrium for every market structure. To demonstrate stability of these equilibria using Lyapunov's indirect method, Lemma 1 provides a decomposition of the Jacobian of the system dynamics that simplify under certain technical assumptions. Lemmas 2-5 then invoke these technical assumptions to characterize an industrial organization network and simplify the expression for the Jacobian of its dynamics. Finally, Lemmas 6, 7, and 9 then yield the machinery to demonstrate how a simple check on the stability of the Grand Structure dynamics will guarantee stability for all other market structures. We now state and prove the stability robustness theorem.

Theorem 1. *Consider an n-product market with agent set $N = \{1, 2, ..., n\}$ and an industrial organization network characterized by (4.4). Let the Grand Coalition, G, of this network be given as in Definition 1, with objective function, U_G, as specified in (4.2). Under these conditions, then (4.4) will have a unique and stable equilibrium for all $\mathscr{F} \in \Delta$, where Δ is the set of all partitions of N, if there exists positive $\varepsilon \in \mathbb{R}$ such that*

$$\max \sigma\left(H(x)\right) \leq -\varepsilon \quad \forall x \in \mathbb{R}^n, \tag{4.22}$$

where $H(x)$ is the Hessian matrix of the objective function $U_G(x)$.

Proof. Let $\mathscr{F}$ be an arbitrary market structure in Δ. Let $J_{V_\mathscr{F}}(x)$ be the Jacobian matrix of $V_\mathscr{F}(x)$ given by (4.4). Following from Lemma 1,

$$J_{V_\mathscr{F}}(x) = \left[A(x) + D_\mathscr{F}\left(A^T(x)\right)\right] + B_\mathscr{F}(x) + C_\mathscr{F}(x).$$

The network structure of demand, $q(x)$, and thus also of utility, $U(x)$, then imply that $B_\mathscr{F}(x) = 0$ as shown in Lemma 4. In the case that $\mathscr{F}$ is the Grand Structure, we know from Lemma 2 that $C_\mathscr{G}(x) = 0$. Thus, $J_{V_\mathscr{G}}(x) = A(x) + A^T(x) = H(x)$. In general, however, we have $J_{V_\mathscr{F}} = A(x) + D_\mathscr{F}(A^T(x)) + C_\mathscr{F}(x)$. From Lemma 6 this yields,

$$W\left(J_{V_\mathscr{F}}(x)\right) = W\left(A(x) + D_\mathscr{F}\left(A^T(x)\right) + C_\mathscr{F}(x)\right)$$
$$\subseteq W\left(A(x)\right) + W\left(D_\mathscr{F}\left(A^T(x)\right)\right) + W(C_\mathscr{F}(x)).$$

From Lemma 9, $W\left(D_\mathscr{F}\left(A^T(x)\right)\right) \subseteq W\left(A^T(x)\right)$, hence

$$W\left(J_{V_\mathscr{F}}(x)\right) \subseteq W\left(A(x)\right) + W\left(A^T(x)\right) + W\left(C_\mathscr{F}(x)\right).$$

As a result,

$$\max \mathfrak{Re} W \left(J_{V_{\mathscr{F}}}(x) \right) \leq \max \mathfrak{Re} W \left(A(x) \right) + \mathfrak{Re} W \left(A^T(x) \right) + \max \mathfrak{Re} W \left(C_{\mathscr{F}}(x) \right).$$

Due to Lemma 5, $W \left(C_{\mathscr{F}}(x) \right) \leq 0$. Also, from Lemma 7,

$$\max \mathfrak{Re} W \left(A(x) \right) + \mathfrak{Re} W \left(A^T(x) \right) = \max W \left(A(x) + A^T(x) \right) = \max W \left(H(x) \right) = \max \sigma(H(x)).$$

Following that, $\max \mathfrak{Re} W \left(J_{V_{\mathscr{F}}}(x) \right) \leq \max \sigma(H(x)) \leq -\varepsilon$. By Corollary 1, we can conclude that the equation $V_{\mathscr{F}}(x) = 0$ has exactly one solution x_e. Hence the market structure $\mathscr{F}$ yields exactly one equilibrium x_e. Moreover, since the Jacobian evaluated at the equilibrium point, $J_{V_{\mathscr{F}}}(x_e)$, satisfies,

$$\max \mathfrak{Re} \sigma(J_{V_{\mathscr{F}}}(x_e)) \leq \max \mathfrak{Re} W(J_{V_{\mathscr{F}}}(x_e)) \leq -\varepsilon < 0,$$

then the equilibrium x_e is locally stable due to Lyapunov's indirect method. $\square$

4.4 Demand Estimation for Industrial Organization Networks

This section shows how we apply the stability robustness condition in Theorem 1 to a class of AIDS-like demand models. We will begin to cover first our main tool, semidefinite programming [11] [5], used in finding the model parameters that best fit the data, while meeting the sufficient condition given in Theorem 1.

4.4.1 Semidefinite Programming

In semidefinite programming, one minimizes a convex function subject to the constraint that an affine combination of symmetric matrices is positive semidefinite. As the authors of [11] noted, such a constraint is nonlinear and nonsmooth, but convex. In fact, it is shown in [11] that although semidefinite programs are much more general than linear programs,

they are not much harder to solve. Most interior-point methods for linear programming have been generalized to semidefinite programs. As in linear programming, these methods have polynomial worst-case complexity, and perform very well in practice.

Let us show the canonical form of a semidefinite program,

$$
\begin{aligned}
\text{minimize} \quad & f_0(x) \\
\text{subject to} \quad & \sigma\left(\Psi_0 + \sum_{i=1}^{n} \Psi_i x_i\right) \leq 0,
\end{aligned}
\tag{4.23}
$$

where $f_0(x)$ is convex and Ψ_i are symmetric for $i = 0, 1, \ldots, n$.

4.4.2 Demand Estimation with Stability Robustness Constraint

Now we will show our methodology applying to a class of demand models. Let us first do so by describing our model, after which we shall show that both the requirements given in Definition 4 and Definition 5 are met. This demand model is based on the concept of effective price: we recognize that changing prices from different price ranges will yield different effects on demand. Therefore, let $f_i(x_i)$ be a function representing the effective price of product i, our demand function will be,

$$
q = Pf(x) + b, \text{where } f(x) = \big(f_1(x_1), f_2(x_2), \ldots, f_n(x_n)\big).
\tag{4.24}
$$

Use demand functions given by Equation (4.24)

Let us show that this demand model given in (4.24) satisfies all the assumptions of an industrial organization network. First, it can be shown that the network assumption in Definition 4 is met, because

$$
q_i(x) = \sum_{j=1}^{n} p_{ij} f_j(x_j).
\tag{4.25}
$$

38

Also the partially convex requirement, as defined in Definition 5, is met. For $j \neq i$,

$$\frac{\partial^2 U_j}{\partial x_i^2}(x) = \frac{\partial^2 [x_j q_j(x)]}{\partial x_i^2} = x_j \frac{\partial}{\partial x_i} \frac{\partial q_j(x)}{\partial x_i} = x_j \frac{\partial}{\partial x_i} \left(p_{ji} \frac{\partial f_j(x_j)}{\partial x_i} \right) = 0 \geq 0. \qquad (4.26)$$

Use splines to design the effective price functions, f(x), in the demand model

These functions should be monotone and will serve as basis functions in a nonlinear regression when fitting P and b from data. The choice of f(x) can be guided by data or use professional expertise to characterize price sensitivity in the market.

Substitute the desired effective price functions to build a semidefinite program. Note that this program samples $H(x)$ to try to enforce that $\sigma(H(x)) \leq -\varepsilon$ everywhere

This is the most important step in the process. Let us be detailed in showing how it is carried out. Assuming that we are given K data points $(\mathbf{q_i}, \mathbf{x_i})$, $i = 1, 2, \ldots K$, where $\mathbf{q_i} \in \mathbb{R}^n$ are quantity demanded at a price setting $\mathbf{x_i} \in \mathbb{R}^n$, our objective is to minimize the regression error. For example, if the regression error is measured by the l_2 norm, then we have a least square regression problem,

$$\text{find} \quad P \in M_n(\mathbb{R}), b \in \mathbb{R}^n \quad \text{to}$$

$$\text{minimize} \quad \sum_{i=1}^{K} \|P\mathbf{x_i} + b - \mathbf{q_i}\|_2^2. \qquad (4.27)$$

In addition, we need to ensure that the condition (4.22) is met. This condition needs to be held for an infinite number of $x \in \mathbb{R}^n$. However, by looking carefully at,

$$H(x) = \frac{\partial^2 U_G}{\partial^2 x} = \left[P \text{diag} \left(\frac{df_i}{dx_i}(x) \right) + \text{diag} \left(\frac{df_i}{dx_i}(x) \right) P^T \right] + \text{diag} \left(\sum_j p_{ij} x_j \frac{d^2 f_i}{dx_i^2}(x) \right),$$
$$(4.28)$$

we recognize that if we require that the effective price functions are linear for $x \in \mathscr{C}_n(R) = \{\mathbb{R}^n, \|x\|_2 > R\}$ for some radius R, then, $H(x)$ is unchanged for $x \in \mathscr{C}_n(R)$. Therefore we

only need to meet the constraint (4.22) for a compact ball $x \in \mathscr{B}_n(R) = \{x \in \mathbb{R}^n \mid \|x\|_2 \leq R\}$. In fact, we will make one step further by sampling the points in this ball, so that the number of points to check is finite. This is often done in practice. So, let $S = \{\mathbf{s_j}\}$ be a finite sample of $x \in \mathscr{B}_n(R)$, constraint (4.22) can be approximated by,

$$\max \sigma\big(H(\mathbf{s_j})\big) \leq -\varepsilon \quad \forall \mathbf{s_j} \in S, \tag{4.29}$$

If we let $\mathbf{y} = \begin{bmatrix} p_{11} & \ldots p_{1n} & b_1 \ldots p_{n1} & \ldots & p_{nn} & b_n \end{bmatrix}^T$, $\Pi \in M_{Kn,n^2+n}(\mathbb{R})$, $\Pi = \mathrm{diag}(\Sigma, \Sigma, \ldots, \Sigma)$, where $\Sigma \in M_{K,n+1}$,

$$[\Sigma]_i = \begin{bmatrix} f_1(x_1) & \ldots f_n(x_i) & 1 \end{bmatrix},$$

$\mathbf{z} = \begin{bmatrix} q_{11} & \ldots & q_{K1} & \ldots q_{1n} & \ldots q_{Kn} \end{bmatrix}^T$, and $l = n^2 + n$, then the regression objective becomes,

$$\text{find} \quad \mathbf{y} \in \mathbb{R}^l \quad \text{to}$$

$$\text{minimize} \quad \|\Pi\mathbf{y} - \mathbf{z}\|_2^2. \tag{4.30}$$

Also, let $\Phi_{ij}(\mathbf{s}) \in M_n(\mathbb{R})$, $\Phi_{ij}(\mathbf{s}) = \mathrm{diag}_{i=1}^n(\sum s_j \frac{\partial^2 f_i}{\partial x_i}(s_i))$, $\Theta_{ij}(\mathbf{s}) \in M_n(\mathbb{R})$ having two non-zero $(i,j)^{th}$ and $(j,i)^{th}$ entries with value $\frac{\partial f_i}{\partial x_i}(s_i)$, $\Psi_{ij} \in M_{n\|S\|}(\mathbb{R})$, $\Psi_{ij} = \mathrm{diag}_{\mathbf{s_k} \in \mathbf{S}} \big[\Phi_{ij}(\mathbf{s_k}) + \Theta_{ij}(\mathbf{s_k})\big]$, and $\Psi_0 \in M_{n\|S\|}(\mathbb{R}), \Psi_0 = \mathrm{diag}(\varepsilon, \varepsilon, \ldots \varepsilon)$, then the regression constraint becomes

$$\text{subject to} \quad \max \sigma \left[\Psi_0 + \sum_{i=1,j=1}^n \Psi_{ij} y_{(n+1)i+j}\right] \leq 0. \tag{4.31}$$

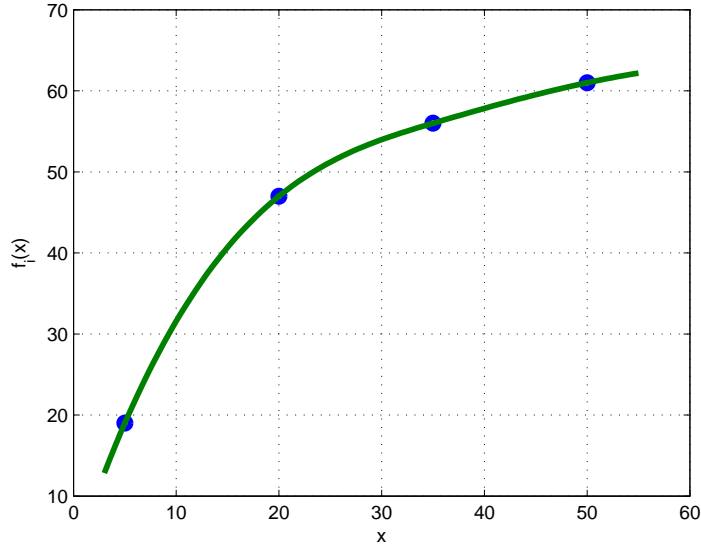(4.30) and (4.31) together constitute a semidefinite program.

Figure 4.1: Plot showing price sensitivity: a spline going through points $(5, 19)$, $(20, 47)$, $(35, 56)$, and $(50, 61)$.

Solve for y - or equivalently - P and b

Solving the least square semidefinite program in (4.30) and (4.31) yields the network demand function, $q(x)$, that best fits the data, and guarantees that the conditions from Theorem 1 on $H(x)$ that guarantee stability robustness for all market structures are met.

4.4.3 Numerical Experiment

Consider 100 data points generated by the log-linear model,

$$\log q(x) = \begin{bmatrix} -0.57 & 0.10 & -0.12 \\ 0.20 & -1.00 & 0.11 \\ -0.02 & 0.06 & -0.68 \end{bmatrix} \log x + \begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix} + w, \tag{4.32}$$

where w is white noise with standard deviation 1. We choose $f_i(\cdot)$ to be the same function for each dimension: a spline going through $(5, 19)$, $(20, 47)$, $(35, 56)$, and $(50, 61)$ (we chose these points by looking at the generated data, and roughly estimating the effects of different price ranges on demand.) A plot showing this spline is shown in Figure 4.1.
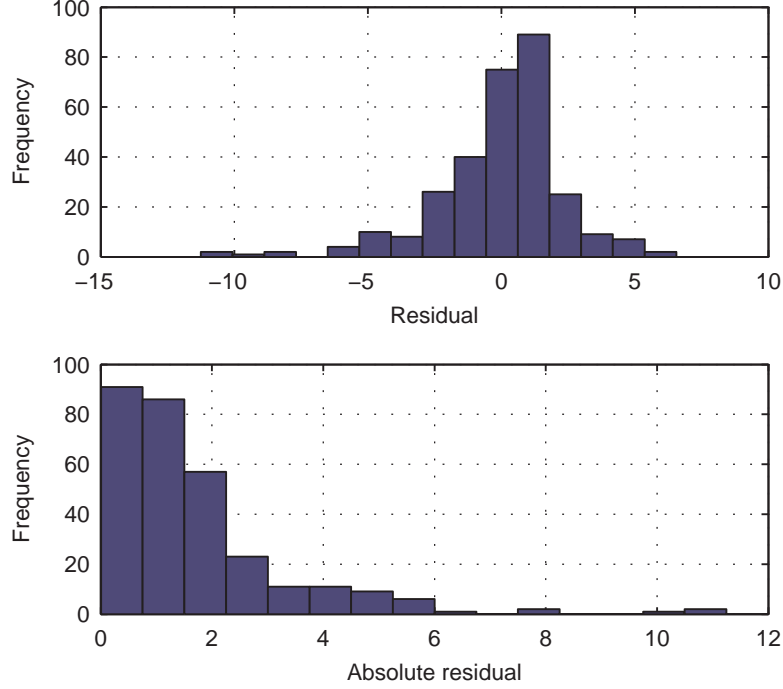
Figure 4.2: Plot showing the histogram of residuals. The top one is the histogram residual values, while the bottom shows a histogram of absolute residuals. These results indicate that the industrial organization network model fits the data extremely well.

Based on this spline, we perform a semidefinite regression to fit the demand function $q = Af(x) + b$ while meeting the robustness condition. The optimal parameters become:

$$q(x) = \begin{bmatrix} -5.70 & 0.96 & -1.23 \\ 1.96 & -10.00 & 1.17 \\ -0.24 & 0.59 & -6.82 \end{bmatrix} f(x) + \begin{bmatrix} 481.22 \\ 636.45 \\ 563.00 \end{bmatrix}. \tag{4.33}$$

These matrices do not look quite the same as the matrices in the original model because our regression model is not in logarithm scale. To see how our model fits the demand, we plot of percentage difference of demands between our regression model and the log-linear model in Figure 4.2. Since we have 100 data points, and each data point reflects the demand of three different products, we show in our plot the histogram of 300 differences, and the histogram of 300 absolute error. While the demanded quantities range between 150 and 400 units, the differences range between 0 and 12 units. For 90% of the data points,

42

the difference is less than 1.5 percent. The maximal difference is about 3.5 percent. Our model fits the data quite well, but more importantly, it guarantees existence, uniqueness, and stability of equilibriums under all market structures.

Note also see that the complimentary/substitutive relationships between different products are also preserved. In the log-linear model, we see that the pairs of products 1 and 2, and 2 and 3 are substitutes, while products 1 and 3 are complements. This is also reflected by the sign of elements of P.

Finally, we show how our demand model reflects own-price demand by plotting q_i with respect to x_i, while fixing both other two prices at 20. The shape looks quite realistic (Figure 4.3), as it shows a decreasing function that gets flatter when price increases, reflecting the law of diminishing returns. These results suggest the method is quite practical.

We also show the log linear demand function in the same plot. The difference between our demand function and the log linear demand function is when price is close to 0, and due to nature of logarithm, log-linear demand increases exponentially fast.

4.5 Conclusion

In this paper we demonstrated stability robustness conditions with respect to coalition structure for a class of profit-maximizing nonlinear systems. These conditions were then leveraged to provide a systematic methodology for estimating a rich variety of demand systems from data that guarantee sensible stability results regardless of the structure of cooperation within the marketplace.

The importance of these results emerges from the ability for regulators and managers alike to reliably conduct market power analyses using merger simulation and reverse merger simulation techniques. In such studies one can compute, for example, the value of cooperation of a firm as a measure of its market power.
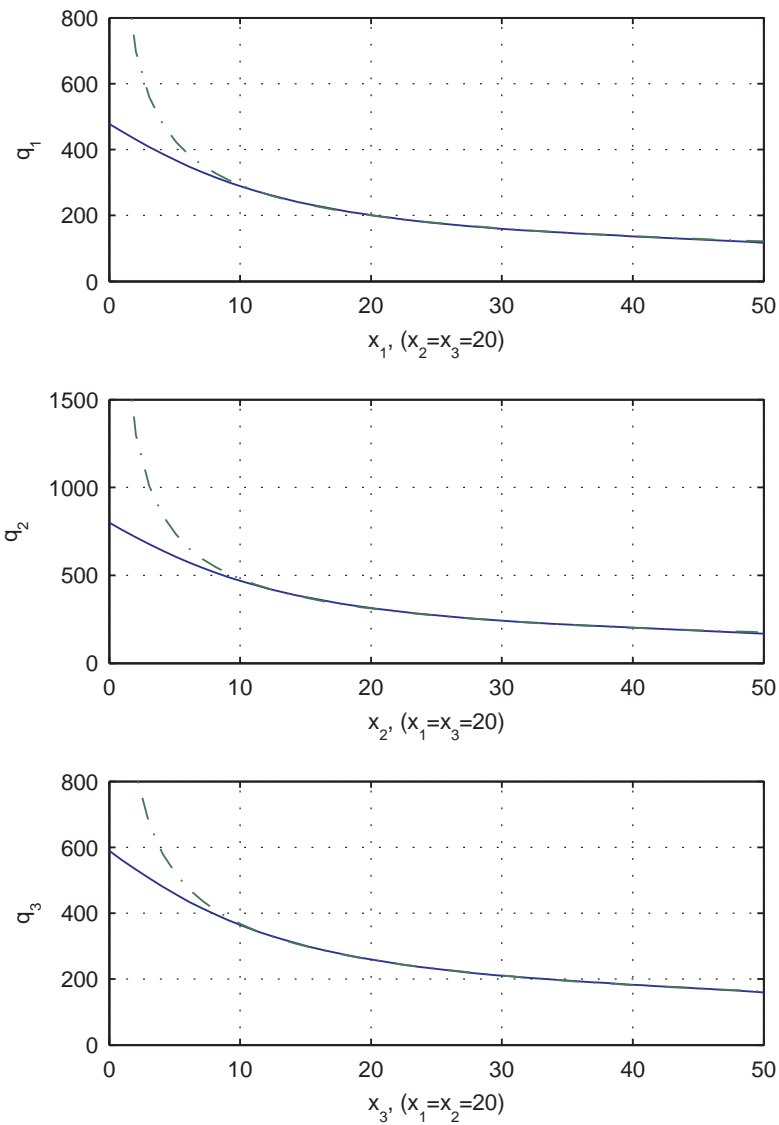
Figure 4.3: Demand plots of each product with respect to its own price, fixing the other two prices at 20. The solid lines plot our demand functions, and the dashed lines plot the loglinear demand functions.

Bibliography

[1] Horizontal merger guidelines, 1997. available at http://www.usdoj.gov/atr/public/guidelines/horiz_book/hmg1.html.

[2] T. R. Beard, G. Ford, and R. Saba. An econometric-driven merger simulation: Considerations and application. *International Journal of the Economics of Business*, 13(2):217–228, July 2006. available at http://ideas.repec.org/a/taf/ijecbs/v13y2006i2p217-228.html.

[3] P. Crooke, L. Froeb, S. Tschantz, and G. Werden. Effects of the assumed demand system on simulated postmerger equilibrium. Papers 97-3, U.S. Department of Justice - Antitrust Division, 1997. available at http://ideas.repec.org/p/fth/usjuat/97-3.html.

[4] K. Gustafson and D. Rao. *Numerical Range*. Springer-Verlag, 1996.

[5] J. Malick. A dual approach to semidefinite least-squares problems. *SIAM J. Matrix Anal. Appl.*, 26(1):272–284, 2005.

[6] D. Rubinfeld and R. Epstein. Merger simulation: A simplified approach with new applications. Competition Policy Center, Working Paper Series CPC01-026, Competition Policy Center, Institute for Business and Economic Research, UC Berkeley, Nov. 2001. available at http://ideas.repec.org/p/cdl/compol/cpc01-026.html.

[7] D. Rubinfeld and R. Epstein. Merger simulation with brand-level margin data: Extending pcaids with nests. Berkeley Olin Program in Law & Economics, Working Paper Series 1088, Berkeley Olin Program in Law & Economics, Aug. 2003. available at http://ideas.repec.org/p/cdl/oplwec/1088.html.

[8] Tran, Giraud-Carrier, Seppi, and Warnick. Cooperation-based clustering for profit-maximizing organizational design. *International Joint Conference on Neural Networks*, pages 1813–17, 2006.

[9] N. Tran, D. West, C. Giraud-Carrier, K. Seppi, S. Warnick, and R. Johnson. The value of cooperation within a profit maximizing organization. *Proceedings of the Fourth*

International Conference on Computational Intelligence in Economics and Finance, pages 1017–20, 2005.

[10] N. Tran, W. S. Weyerman, C. Giraud-Carrier, K. Seppi, S. Warnick, and R. Johnson. Studies in the dynamics of economics systems. *Proceedings of the 2005 IEEE Conference on Control Applications*, pages 861–866, 2005.

[11] L. Vandenberghe and S. Boyd. Semidefinite programming. 38(1):49–95, Mar. 1996.

[12] R. P. S. V.C. Prasad. Sufficient conditions for existence and diffeomorphism of a nonlinear vector function. *Proceedings of the IEEE*, 75(6):846–7, 1987.

[13] G. Werden, G.J., and L. Froeb. Simulation as an alternative to structural merger policy in differentiated products industries. Papers 95-02, U.S. Department of Justice - Antitrust Division, 1995. available at http://ideas.repec.org/p/fth/usjuat/95-02.html.