Stability Robustness Conditions for Gradient Play Differential Games with Partial Participation in Coalitions

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Abstract— This paper formulates and solves the stability robustness problem for a class of gradient play cooperative differential games with coalition structures allowing for partial participation and symmetric contracts. We show that when the fully cooperative grand coalition generates stable dynamics, then all other coalition structures are stable. This stability guarantee is important in designing localized cooperative dynamics satisfying global objectives in large-scale distributed systems.

I. INTRODUCTION: COMPETITION AND COOPERATION

Market dynamics, governed by competition, are one of the most well-studied multi-agent systems. Firms, as coalitions of agents, cooperate in the hopes of maximizing profit by offsetting the computational limitations of individual agents, better competing for scarce resources. Often it is competition among agents that drives groups of individuals to organize into coalitions and cooperate to form a more viable competitive force. In many situations, interactions between agents can be a mixture of both cooperative and non-cooperative behavior. With existing game theory models however, partial cooperation is often difficult to manage.

In the last few decades, game theory has shed light on the rational behavior of strategies and decision-making. Agents often share both common and conflicting interests, making the balance between cooperation and competition difficult to obtain. This paper focuses on the stability of systems allowing partial cooperation between agents. Agents compete, with the self-interested goal of maximizing their own payoff, but through the value of cooperation, may find that this is best done by agreeing to help others.

The issue of robustness to cooperative effects by allowing for perfectly cooperative coalitions was addressed previously [10]. The resulting system was competitive between coalitions but cooperative within coalitions. Under certain broad conditions, if the system is stable under a single coalition, it will remain stable for all other coalition structures. This work extends these previous stability robustness results to situations where coalitions may not be perfectly cooperative, but instead represent partial cooperation between agents. Again, we demonstrate computable stability robustness conditions.

The next section begins with a review of essential game theory concepts. Section III introduces the class of games considered and Section IV formulates the stability robustness problem. Section V provides stability robustness conditions to guarantee no partially cooperative coalition structure will destabilize the system. Finally, Section VI illustrates the ideas with an example and discusses the result's importance.

II. BACKGROUND: GAME THEORY

Game theory has taken great strides to mathematically model competition and cooperation [6]. Beginning with von Neumann's classic work [11], *zero-sum games* have been explored, where a player's win implies his competitor's loss [2]. *Non-zero-sum games* allow for non-zero aggregate losses or gains, as described by a payoff function. Thus players cooperate in the hopes of achieving a win-win situation.

Repeated games, or *dynamic games*, characterize games played repeatedly on an infinite time horizon. No move is ever necessarily the last move, and a player must consider the effect his move will have on the future [7]. Strategies therefore reflect the long term relationship between players.

In *infinite dynamic games*, there are an infinite number of alternatives in the action sets of players. To find solutions, one cannot follow the branches of a finite tree structure, but must instead find solutions to state equations that describe the evolution of the decision process [1]. A *differential game* is a continuous-time infinite dynamic game, where the state evolution is described by a differential equation, which depends on the strategy of each player [5]. In *gradient play* differential games, a player's strategy is characterized by the gradient of his payoff function [9], [3].

In *noncooperative games*, players cannot form binding agreements to cooperate with others, but instead act alone. In *cooperative games*, players can enter into agreements, cooperating to achieve objectives. Although cooperative theory is relevant for differential games, it is far less developed and studied. Cooperative games allow for the coalition formation, and are a competition between coalitions as opposed to competition between individual agents. This competition between coalitions leads to payoffs for coalitions and not for individual players. Under the assumption of transferable utility between agents, a *payoff function* describes how earnings will be distributed among the agents in a coalition.

In this paper we deal with non-zero-sum, gradient play differential games. Our formulation differs from standard cooperative game theory in that we consider a situation where the payoff to one coalition depends on the coalition structure of the remaining agents. We will describe this situation by using the term *differential games with coalitions*.

III. DIFFERENTIAL GAMES WITH COALITIONS

In this work, an "agent" is an autonomous and sovereign decision maker. We assume that the set of actions available to each agent is represented by $x_i(t) \in \mathbb{R}$, $i = 1, 2, ..., n$.

A. Profit Function

Each agent is characterized by a profit function that may depend on the actions of other agents, and a decision process that determines how the agent will choose actions over time. We restrict ourselves to the class of profit functions given by

$$\Pi_i(x) = x_i q_i(x). \tag{1}$$

One interpretation is to let x_i be the mark-up price of product i, controlled by agent i, with $q_i(x)$ as a demand function for product i describing how the quantity sold depends on price.

B. Decision Process

An agent's decision process, f_i, is a function that maps information available to the agent, such as its profits at any given time, $\Pi_i(x(t))$, the actions of the other agents, $x(t)$, etc. to the derivative of the agent's action. Collectively, these processes then characterize the dynamics of the game as

$$\dot{x}_i = f_i(x), \ i = 1, \dots, n.$$

We assume that agents have full state information of each other's actions and can measure $x(t)$ at every time instance.

First we consider the situation where agents seek to maximize their own profit function. One strategy is to choose a decision process that is the gradient of its profit function:

$$\dot{x}_i = f_i(x) = \frac{\delta \Pi_i}{\delta x_i} = q_i(x) + x_i \frac{\delta q_i}{\delta x_i}, \ i = 1, \dots, n.$$

Restricting profit functions to that of (1) gives the autonomous dynamic system describing the game's dynamics:

$$\dot{x} = diag(J_\Pi(x)) = \left(diag(J_q^T(x))\right)x + q(x) \tag{2}$$

where $\Pi(x) = [\Pi_1(x) \dots \Pi_n(x)]^T$, $J_{f(x)}(x)$ is the Jacobian of a function $f(x)$ with respect to x, and $diag(M)$ is the vector of the diagonal elements of a matrix $M \in \mathbb{R}^{n \times n}$.

C. Coalitions and Coalition Structure

The dynamics specified by (2) give a non-cooperative non-zero-sum game, where each agent maximizes his own profit without regard to others. Interaction between agents arises only as a result of coupling in the demand function, $q(x)$, if, e.g., their products as complements in the demand function.

In this work, *cooperation* is an explicit, externally enforceable contract between agents, that changes the decision processes of the cooperating parties to consider the interest of the group and not only self-interest. The contract specifies a payoff to each group member with the cooperating parties forming a *coalition*. The *coalition structure* of the game is the set of all coalitions. We assume the coalition structure is decided before the game begins and that it remains constant.

For example, consider a game with four players and two coalitions. Many ways of creating two coalitions from four players exist, and each defines a coalition structure. One possibility is $\{(1,2,3),(3,4)\}$, where players 1, 2, and 3 form a coalition, and players 3 and 4 form another. Alternatively, $\{(1,2,3),(4)\}$ describes a coalition structure where players 1, 2, and 3 form a coalition, and player 4 creates his own "coalition." We would expect each of these structures to yield different game dynamics for the same demand function.

D. Fully Cooperative Coalitions

The simplest type of coalition is a *fully cooperative* coalition, where members of the coalition cooperate with each other only and there is a partition on the set of players. We can encode this structure with a matrix $P = [p_{ij}] \in \mathbb{R}^{n \times n}$ where $p_{ij} = 1$ if players i and j are in the same coalition, and 0 otherwise. This P is called the *participation matrix* and completely captures the coalition structure of the game.

Given $\Pi(x)$ and P, the players' payoff can be characterized by a *distribution matrix* $D = [d_{ij}] \in \mathbb{R}^{n \times n}$. This positive definite, diagonal matrix specifies a percentage of the coalition profits awarded to each player. The payoffs, $V(x)$, are

$$V(x) = DP\Pi(x). \tag{3}$$

Example 1: Consider a game with coalition structure $\{(1,2),(3)\}$ and a distribution matrix D, so that

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Suppose that at a given point in time, $\Pi(x(t)) = [20, 40, 30]^T$. The payoffs to each player at that time are then given by

$$V(x) = DP\Pi(x) = \begin{bmatrix} 30 & 30 & 30 \end{bmatrix}^T.$$

Characterizing the payoff to each player allows us to define the dynamics for the game. In the gradient play situation, each player will adjust his actions in the direction of his payoff rather than the gradient of his profit. This leads to

$$\dot{x}_i = f_i(x) = \frac{\delta V_i}{\delta x_i}, \ i = 1, \dots, n.$$

for each agent, and the entire vector $\dot{x}$ is given by

$$\begin{aligned} \dot{x} = \frac{\delta V}{\delta x} &= diag(DPJ_\Pi(x)) \\ &= \left((DP) \circ J_q^T(x)\right)x + Dq(x) \\ &= D\left((P \circ J_q^T(x))x + q(x)\right) \end{aligned} \tag{4}$$

where the symbol $\circ$ represents the Hadamard product, or pointwise multiplication of any two matrices of identical size.

We assume that agents know the partial derivatives of all other agents' profit functions with respect to themselves. In other words, agent i knows $\frac{\delta \Pi_k}{\delta x_i}$ for all agents k (with whom i is cooperating). Based on their relationship, it is reasonable to

assume such information might be shared. Comparing equations (2) and (4) we see the explicit impact of cooperation contracts on the dynamics of the game operating through the participation matrix P and the distribution matrix D.

Note that the number of fully cooperative coalition structures available in a game with n players is large, but finite, given by the Bell number. Let $\mathscr{F}$ be the set of participation matrices describing full cooperation among competing agents. For example, in the case when there are three agents, the set of full cooperation matrices, $\mathscr{F}_3$, is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

In any set $\mathscr{F}$, there is a *grand coalition* P_G characterized as

$$P_G = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \tag{5}$$

where every agent belongs to a single coalition. From (4), it is clear that each of these coalition structures will yield a different dynamic system characterizing the game.

A given distribution matrix D is considered admissible if

$$\|V(x^*)\|_1 = \sum_{i=1}^{n} d_{ii}(P\Pi(x^*))_i = \|\Pi(x^*)\|_1$$

holds at equilibrium, x^*. In other words, profit, e.g. money, is not created or lost at equilibrium. Note that for full cooperation matrices, D can be chosen so that $\|V(x^*)\|_1 = \|\Pi(x^*)\|_1$ holds at all points in time, since, for example, D can be chosen such that DP is stochastic.

E. Partially Cooperative Coalitions

Sometimes players may find themselves in the position of participating in more than one coalition at the same time. In these situations, the player has limited attention that he must split between these coalitions. This implies that the player is only partially participating in the coalitions he has joined.

Absolute full cooperation among all agents is specified using the grand coalition. In that case, everyone equally values the profit of all agents. The opposite case is when everyone works exclusively by themselves, and P is given by the identity matrix. The convex combinations of the matrices F allow for cooperation to range between these two extremes.

This splitting of a player's loyalties can be described through the participation matrix, P, where p_{ij} is allowed to vary based on the degree of loyalty player i has with player j. We restrict ourselves to *symmetric contracts*, where $p_{ij} = p_{ji}$. This leads us to extend the definition of P as follows:

$$\begin{aligned} p_{ij} &= 1 & \text{if } i = j \\ 0 \leq p_{ij} &\leq 1 & \text{if } i \neq j \\ p_{ij} &= p_{ji} & \forall i, j \end{aligned} \tag{6}$$

Each entry p_{ij} is the degree that agent i values the profit of agent j, compared to how the agent values its own profit, when choosing how to adjust its actions, x_i. Furthermore, an agent will never value the profit of another agent more
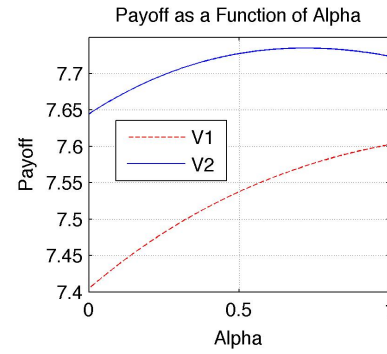

Fig. 1. The payoff for agents varies as α ranges from 0 to 1.

than it values its own, as all agents are considered to be motivated by self-interest. Therefore $p_{ij} \leq 1$. Nevertheless with this extended definition of the participation matrix P, the dynamics of the game, given by (4), remain the same.

When $p_{ij} > \varepsilon$, where ε is a small number, in the participation matrix, then agents i and j have agreed to form a coalition. The process by which agents establish the values of the participation matrix is not discussed here. In general though, one may assume that there are some set of dynamics governing how the values of P change over time.

Example 2: Consider a simple market with only 2 agents. Let their respective demand, profit, and payoff functions be:

$$\begin{bmatrix} q_1(x) \\ q_2(x) \end{bmatrix} = \begin{bmatrix} -3 & -0.9 \\ -0.8 & -2.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 11 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = DP\Pi(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix} \begin{bmatrix} \Pi_1 \\ \Pi_2 \end{bmatrix}$$

where α is the relative cooperation between Agents 1 and 2 in the participation matrix P. Figure 1 shows how the payoff, $V(x)$, varies for both Agents 1 and 2 at equilibrium as their relative participation with each other, α, varies from 0 to 1. One can see that if the distribution function D is the identity matrix, then Agent 2 will want to cap cooperation at $\alpha = 0.7$ as its payoff will begin to decrease if relative cooperation with Agent 1, α, is raised. Note however that the total value of the coalition, $V_1(x) + V_2(x)$ continues to increase as α goes to 1. Therefore if a different distribution matrix D were defined, where Agent 1 shares some of its marginal payoff with Agent 2, it would be advantageous for both to fully cooperate, i.e. $\alpha = 1$. One can thus see that the optimal level of cooperation between agents depends not only on the total payoff to the coalition, but also on how the payoff is distributed, based on the choice of D.

IV. PROBLEM FORMULATION

Equation (4) specifies a rich class of partially cooperative differential games. The resulting dynamics can be quite nonlinear and difficult to analyze, in general. Moreover, there can be significant uncertainty about the actual cooperation structure governing the dynamics of a game. This problem seeks computable conditions that guarantee the stability of the game with respect to uncertain coalition structure.

Problem 1: (Stability Robustness with Respect to Partially Cooperative Coalitions) Given a profit function $\Pi(x)$ as in (1), find conditions when the gradient play differential game with partial participation in coalitions, as characterized by (4), has a unique stable equilibrium for each admissible contract, characterized by a participation matrix P satisfying (6) and a positive definite diagonal distribution matrix D.

V. STABILITY ROBUSTNESS CONDITIONS

The solution to this problem leverages a number of results that are included as lemmata. In the end, we will see that certain broad conditions on the allowable profit functions, $\Pi(x)$, will enable a simple check for stability robustness by checking the stability of the grand coalition. Lyapunov's indirect method will be applied to determine if the differential equation given in (4) reaches a stable steady state.

Lemma 1: If a square matrix $F \in \mathscr{F}_n$, for any $n \in \mathbb{N}$, then F is positive semi-definite.

Proof: All matrices $F \in \mathscr{F}_n$, for any $n \in \mathbb{N}$ can be constructed so that $F = g^T g$, where g is a matrix of size $r \times n$, where $r = rank(F)$, and n is the size of the square matrix F. g is comprised of the r distinct rows of F. By factoring the matrix F in this way, it is easy to see that F has $n - r$ eigenvalues of value 0 and the remaining r eigenvalues of F are the sums of each of the rows of g, which may or may not be distinct values, but are all positive values. ∎

Example 3: Let

$$F = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then $g = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ and $\sigma(F) = \{2,3,1,0,0\}$ are the eigenvalues of F.

Lemma 2: For matrix $A \in \mathbb{R}^{n \times n}$, if $A = \alpha_1 F_1 + \cdots + \alpha_k F_k$, where $F_i \in \mathscr{F}_n$, $\alpha_i \geq 0 \ \forall i = \{1, \ldots, k\}$, and $\sum_{i=1}^{k} \alpha_i = 1$, then A is positive semi-definite (PSD).

Proof: By Lemma 1, all $F \in \mathscr{F}$ are PSD. Any PSD matrix multiplied by a non-negative scalar is PSD and the sum of PSD matrices is PSD. Convex combinations of full cooperation matrices are therefore PSD. ∎

We now focus on proving the stability of (4), which was

$$\dot{x} = f(x) = diag(DPJ_\Pi(x)).$$

In doing so, we first define the Jacobian of this system, which will later be used in proving local stability. Stability will first be proven when D is the identity matrix, and then using a Lyapunov argument, stabilty is proven when D is included.

Lemma 3: When $D = I$, the Jacobian matrix, R, of the system in (4) can be written as

$$R = \frac{\delta f}{\delta x} = (diag(P))A + P \circ A^T + B \qquad (7)$$

where the matrices $A, B \in \mathbb{R}^{n \times n}$ are given as:

$$A_{ii} = \frac{1}{2} \sum_{k=1}^{n} \frac{\delta \Pi_k^2}{\delta x_i \delta x_i}, \qquad A_{ij} = \frac{\delta \Pi_i^2}{\delta x_i \delta x_j}$$

$$B_{ij} = 0, \qquad B_{ii} = \sum_{k=1}^{n} (p_{ik} - p_{ii}) \frac{\delta \Pi_k^2}{\delta x_i \delta x_i} \qquad (8)$$

Proof: Let $R = \frac{\delta f}{\delta x}$. The diagonal elements of R are therefore given by:

$$R_{ii} = \frac{\delta f_i}{\delta x_i} = \frac{\delta}{\delta x_i} \sum_{k=1}^{n} [p_{ik} \frac{\delta \Pi_k}{\delta x_i}] = \sum_{k=1}^{n} p_{ik} \frac{\delta \Pi_k^2}{\delta x_i \delta x_i}$$

Likewise, the off-diagonal elements of R are given by

$$R_{ij} = \frac{\delta f_i}{\delta x_j} = \frac{\delta}{\delta x_j} \sum_{k=1}^{n} [p_{ik} \frac{\delta \Pi_k}{\delta x_i}] = \frac{\delta}{\delta x_i} \sum_{k=1}^{n} p_{ik} \frac{\delta \Pi_k^2}{\delta x_i \delta x_j}$$

which can be simplified to

$$R_{ij} = p_{ii} \frac{\delta \Pi_i^2}{\delta x_i \delta x_j} + p_{ij} \frac{\delta \Pi_j^2}{\delta x_i \delta x_j}$$

as $\frac{\delta \Pi_k^2}{\delta x_i \delta x_j}$ is only non-zero when $k \in \{i, j\}$, where i, j are the indices of the off-diagonal entries in R_{ij}. Furthermore, let

$$R_{ii} = \sum_{k=1}^{n} p_{ii} \frac{\delta \Pi_k^2}{\delta x_i \delta x_i} + \sum_{k=1}^{n} (p_{ik} - p_{ii}) \frac{\delta \Pi_k^2}{\delta x_i \delta x_i}$$

∎

Definition 1: A profit function $\Pi(x) : \mathbb{R}^n \to \mathbb{R}^n$ is said to be *partially convex* if

$$\frac{\delta^2 \Pi_j}{\delta x^2}(x) \geq 0 \quad \forall j \neq i, \ \forall x \in \mathbb{R}^n.$$

Lemma 4: The matrix B, as given in Lemma 3 is a negative semi-definite diagonal matrix when the profit functions of the system are partially convex and P is as defined in (6).

Proof: Note that B is a diagonal matrix and so it will be negative semi-definite (NSD) if all of its diagonal entries are non-negative, as these entries are also its eigenvalues. By the definition of partial convexity, $\frac{\delta^2 \Pi_j}{\delta x^2}(x) \geq 0$. From (6), we see that $p_{ik} - p_{ii} \leq 0 \ \forall i, k \in \{1, \ldots, n\}$. Therefore B is NSD. ∎

Lemma 5: $B(x) = 0$ when considering the grand coalition.

Proof: In the case of the grand coalition, P is the ones matrix, as given in (5), and so $p_{ik} = p_{ii}, \forall i, k$. Then since B is a diagonal matrix with diagonal entries of

$$B_{ii} = \sum_{k=1}^{n} (p_{ik} - p_{ii}) \frac{\delta U_k^2}{\delta x_i \delta x_i},$$

then B is a zero-matrix and $R = A + A^T$.

We now give various results about numerical range, W. Let $\sigma(A)$ denote the eigenvalues of A and let $co(A)$ denote the convex hull of A. The following lemma is from [4]:

Lemma 6: Let $A, B \in \mathbb{C}^{n \times n}$.

(i) $W(A)$ is compact and convex.
(ii) $co(\sigma(A)) \subseteq W(A)$.
(iii) $W(A + B) \subseteq W(A) + W(B)$.
(iv) A is normal $\Rightarrow co(\sigma(A)) = W(A)$.

The next lemma is from [10] and Lemma 8 is directly from [8]. The symbol $\Re X$ denotes the real part of X.

Lemma 7: For $A \in \mathbb{R}^{n \times n}$,

$$\max W(A + A^T) = \max \Re W(A) + \max \Re W(A^T).$$

Lemma 8: Given $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(x) = y$ will have exactly one root for each y if there exists positive $\varepsilon \in \mathbb{R}$ such that

$$\max \Re W\left(\frac{\partial f}{\partial x}(x)\right) \leq -\varepsilon \quad \forall x \in \mathbb{R}^n.$$

The next result shows that when any PSD P with 1s on the diagonal is pointwise-multiplied with any square matrix A of the same size, the numerical range of $P \circ A$ is contained within the numerical range of A. More precisely we have:

Lemma 9: If $P = Q^*Q \in \mathbb{R}^{n \times n}$ where $Q \in \mathbb{C}^{m \times n}$ having $p_{ii} = 1$, then with any square $A \in \mathbb{R}^{n \times n}$, $W(P \circ A) \subseteq W(A)$.

Proof: Let $M = \{1, \ldots, m\}$, and $N = \{1, \ldots, n\}$. Let $B = P \circ A \Rightarrow b_{ij} = p_{ij}a_{ij} = \sum_{k \in M} \overline{q_{ik}} q_{jk} a_{ij}$. For any $w \in W(B) = W(P \circ A)$ we want to prove that $w \in W(A)$. Since $w \in W(B)$, there exists a unit vector $\mathbf{x}$ such that

$$w = \mathbf{x}^* B \mathbf{x} = \sum_{i,j \in N} \overline{x_i} b_{ij} x_j = \sum_{i,j \in N} \overline{x_i} \Big[\sum_{k \in M} \overline{q_{ik}} q_{jk} a_{ij} \Big] x_j$$

$$= \sum_{k \in M} \sum_{i,j \in N} \overline{x_i} \, \overline{q_{ik}} q_{jk} a_{ij} x_j = \sum_{k \in M} \sum_{i,j \in N} [\overline{x_i} \, \overline{q_{ik}}] a_{ij} [q_{jk} x_j]. \quad (9)$$

Let $\mathbf{y_k} \in \mathbb{R}^n$, $[\mathbf{y_k}]_i = q_{ik} x_i$, and let $L = \{k \in M : \mathbf{y_k} \neq 0\}$, then

$$\sum_{k \in L} \|\mathbf{y_k}\|^2 = \sum_{k \in M} \|\mathbf{y_k}\|^2 = \sum_{k \in M} \sum_{i \in N} [\mathbf{y_k}]_i^2 = \sum_{k \in M} \sum_{i \in N} m_{ik}^2 x_i^2$$

$$= \sum_{i \in N} x_i^2 \Big[\sum_{k \in M} m_{ik}^2 \Big] = \sum_{i \in N} x_i^2 p_{ii} = \sum_{i \in N} x_i^2 = \|x\|^2 = 1.$$

Substitute $\mathbf{y_k}$ into equation (9),

$$w = \sum_{k \in M} \sum_{i,j \in N} \overline{[\mathbf{y_k}]_i} a_{ij} [\mathbf{y_k}]_j = \sum_{k \in M} \mathbf{y_k}^* A \mathbf{y_k} = \sum_{k \in L} \mathbf{y_k}^* A \mathbf{y_k}$$

$$= \sum_{k \in L} \|\mathbf{y_k}\|^2 \frac{\mathbf{y_k}^*}{\|\mathbf{y_k}\|} A \frac{\mathbf{y_k}}{\|\mathbf{y_k}\|} \in \mathrm{co}\Big\{ \frac{\mathbf{y_k}^*}{\|\mathbf{y_k}\|} A \frac{\mathbf{y_k}}{\|\mathbf{y_k}\|} \Big\}_{k \in L} \subseteq W(A).$$

∎

We now prove that the system for $\dot{x}$ in (4) has a stable equilibrium for any matrix P given that P is PSD and satisfies (6), when $D = I$, if the system is stable for the grand coalition. Therefore, in order to determine if any cooperation structure will be stable for a given profit function, one need only determine if the system is stable for the grand coalition cooperation structure. Stability for the grand coalition implies stability for any other P satisfying (6).

Theorem 1: The n-player game given as in (4), with $D = I$, has a unique and stable equilibrium for all PSD P satisfying the conditions in (6), if there exists $\varepsilon > 0 \in \mathbb{R}$ such that

$$\max \sigma(R_G(x)) \leq -\varepsilon \quad \forall x \in \mathbb{R}^n,$$

where $R_G(x)$ is the Jacobian matrix of the system dynamics, as in (7), for P corresponding to the grand coalition.

Proof: Let P be an arbitrary cooperation matrix satisfying (6) and being PSD. Following from Lemma 9,

$$R = (diag(P))A + P \circ A^T + B$$

When P is the grand coalition, we know from Lemma 5 that $B_G = 0$. Thus, $R_G = A(x) + A^T(x)$. In general, however, we

have $R = A(x) + P \circ A^T(x) + B(x)$ since P is restricted to have value 1 on the diagonal. From Lemma 6 this yields,

$$W(R(x)) = W\big(A(x) + P \circ A^T(x) + B(x)\big)$$
$$\subseteq W(A(x)) + W\big(P \circ A^T(x)\big) + W(B(x)).$$

From Lemma 9, $W\big(P \circ A^T(x)\big) \subseteq W\big(A^T(x)\big)$, hence

$$W(R(x)) \subseteq W(A(x)) + W\big(A^T(x)\big) + W(B(x)),$$

$$\max \Re W(R(x)) \leq \max \Re W(A(x)) + \Re W\big(A^T(x)\big)$$
$$+ \max \Re W(B(x)).$$

Due to Lemma 5, $W(B(x)) \leq 0$. Also, from Lemma 7,

$$\max \Re W(A(x)) + \Re W\big(A^T(x)\big) = \max W\big(A(x) + A^T(x)\big)$$
$$= \max W\big(R_G(x)\big) = \max \sigma(R_G(x)).$$

Following that, $\max \Re W(R_G(x)) \leq \max \sigma(R_G(x)) \leq -\varepsilon$. By Lemma 8, we can conclude that the equation $f(x) = 0$ has exactly one solution x^*. Hence the cooperation matrix P yields exactly one equilibrium x^*. Moreover, since the Jacobian evaluated at the equilibrium point, $f(x^*)$, satisfies,

$$\max \Re \sigma(f(x^*)) \leq \max \Re W(f(x^*)) \leq -\varepsilon < 0,$$

then the equilibrium x^* is locally stable due to Lyapunov's indirect method. ∎

Theorem 2: The n-player game given as in (4), with any diagonal positive definite D, will have a unique and stable equilibrium for all positive semi-definite P satisfying the conditions in (6), if there exists positive $\varepsilon \in \mathbb{R}$ such that

$$\max \sigma(R_G(x)) \leq -\varepsilon \quad \forall x \in \mathbb{R}^n,$$

where $R_G(x)$ is the Jacobian matrix of the system dynamics, as in (7), for P corresponding to the grand coalition.

Proof: Let $\hat{P} = DP$. Using the general form of the Jacobian matrix R of the system (4), as described in Lemma 3, where $R = (diag(\hat{P}))A + \hat{P} \circ A^T + \hat{B}$, then let us define

$$C = diag(DP)A + DP \circ A^T + B$$
$$= DA + DP \circ A^T + B = DA + P \circ DA^T + B$$
$$C^T = (diag(DP)A)^T + P^T D^T \circ A + B$$
$$= A^T D + PD \circ A + B = A^T D + P \circ AD + B$$

Let $X = D^{-1} > 0$, and define $Q = C^T X + XC$. Then

$$C^T X = A^T D D^{-1} + (P \circ AD)D^{-1} + BD^{-1}$$
$$= A^T + P \circ A + BD^{-1}$$
$$XC = D^{-1} DA + D^{-1}(P \circ DA^T) + D^{-1} B$$
$$= A + P \circ A^T + D^{-1} B$$
$$Q = (A + A^T) + P \circ (A + A^T) + D^{-1} B + BD^{-1}.$$

Note that B is NSD using the definition (8). Multiplication by positive D cannot affect the sign of its entries. Therefore, $D^{-1} B$ and BD^{-1} are NSD. Also, $A + A^T$ is negative definite and $P \circ (A + A^T)$ is NSD. Therefore Q is negative definite, which proves that the matrix $DA + (DP) \circ A^T + B$ is stable. ∎

VI. Example

We now provide an example to show the stability robustness results. We will use the dynamics of $V(x) = DP\Pi(x)$ where $\Pi_i(x) = q_i(x)x_i$ and $\dot{x}_i = \frac{\delta V_i(x)}{\delta x_i}$ as defined previously.

First let us consider a market consisting of three products with a linear consumer demand function, $q(x)$, defined as

$$\begin{bmatrix} q_1(x) \\ q_2(x) \\ q_3(x) \end{bmatrix} = \begin{bmatrix} -8 & 1 & -5 \\ 4 & -4 & -6 \\ -2 & -8 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 70 \\ 90 \\ 170 \end{bmatrix}$$

Note that based on the signs of the coefficients in the demand function, we see that Products 1 and 2 have a substitutive relationship, while Products 1 and 3, along with 2 and 3, have a complementary relationship. That is to say, an increase in the price of Product 1 results in decreased sales of both Products 1 and 3, but an increase of sales of Product 2.

Now consider a participation P and distribution D where

$$P = \begin{bmatrix} 1 & 0.2 & 0 \\ 0.2 & 1 & 0.3 \\ 0 & 0.3 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 0.8 & 0 & 0 \\ 0 & 0.75 & 0 \\ 0 & 0 & 0.7 \end{bmatrix}$$

This indicates that Agents 1 and 3 are not cooperating, Agents 1 and 2 consider the profits of the other at 20% of how they consider their own profits when determining prices, and Agents 2 and 3 consider the profits of the other at 30% of how they regard themselves. D was chosen to guarantee that money is neither lost or gained in the system at equilibrium.

From (4), we find that

$$\dot{x} = \begin{bmatrix} -12.8 & 1.44 & -4 \\ 3.15 & -6 & -6.3 \\ -1.4 & -6.86 & -12.6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 70 \\ 90 \\ 170 \end{bmatrix}$$

corresponding to equilibrium values of $x^* = \{3.33, 6.93, 9.35\}$ dollars, a demand of $q(x^*) = \{3.57, 19.5, 23.77\}$ units sold, and total profit and payoff for all agents combined being $\|\Pi(x^*)\|_1 = \|V(x^*)\|_1 = 369$. This system is stable, but this information is not enough to determine if the system will be stable for any coalition structure P. According to Theorem 2, the system is only guaranteed to be stable for all possible coalition structures if it is stable for the grand coalition. In fact, this system is unstable for the grand coalition, and so any of the possible P matrices may or may not be stable; nothing is guaranteed.

For example, the system is unstable for

$$P = \begin{bmatrix} 1 & 0.4 & 0.9 \\ 0.4 & 1 & 0.3 \\ 0.9 & 0.3 & 1 \end{bmatrix}.$$

The trajectories of its unstable prices are shown in Figure 2.

However, now consider the demand function $q(x)$ of

$$\begin{bmatrix} q_1(x) \\ q_2(x) \\ q_3(x) \end{bmatrix} = \begin{bmatrix} -6 & -5 & 4 \\ -4 & -8 & 3 \\ 1 & 2 & -15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 30 \\ 40 \\ 25 \end{bmatrix}$$

showing that Products 1 and 2 are imperfect complements, whereas Products 2 and 3, and also 1 and 3, are imperfect substitutes. Now let P be the grand coalition and let D be

$$D = \begin{bmatrix} 0.70 & 0 & 0 \\ 0 & 0.65 & 0 \\ 0 & 0 & 0.70 \end{bmatrix}$$
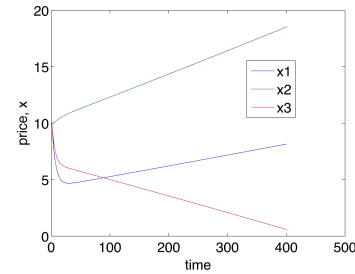

Fig. 2. This system is unstable and so prices never equilibrate.

Then we find that the system has dynamics

$$\dot{x} = \begin{bmatrix} -8.4 & -6.3 & 3.5 \\ -5.85 & -10.4 & 3.25 \\ 3.5 & 3.5 & -21 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 30 \\ 40 \\ 25 \end{bmatrix}$$

which is stable and gives equilibrium values $x^* = \{1.84, 3.45, 2.07\}$ dollars and a demand of $q(x^*) = \{9.95, 11.19, 2.65\}$ units. By Theorem 2, any other valid coalition structure will also be stable, e.g.

$$P = \begin{bmatrix} 1 & 0.8 & 0.3 \\ 0.8 & 1 & 0.4 \\ 0.3 & 0.4 & 1 \end{bmatrix}.$$

VII. Conclusion

This paper introduced a differential game with partial participation in coalitions based on gradient play parameterized by a participation matrix P and a distribution matrix D that cover a wide class of cooperative dynamics. We formulated the problem of finding sufficient conditions to guarantee stability for all coalition structures. We then proved stability robustness with respect to coalition structure for this class of differential games. This result extends previous results to account for partial cooperation in coalition structures and demonstrates that when the system is stable for the grand coalition, then all other coalition structures are also stable.

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