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Target control and source estimation metrics for dynamical networks[☆]

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ABSTRACT

This note examines metrics for target control and source estimation in dynamical networks. Specifically, the energy required to control a target node in a network from a remote input, and dually the fidelity with which a source state can be estimated from a noisy remote measurement, are studied. For both problems, spectral and graph-theoretic analyses are undertaken, and a comparison between the two metrics is also developed. The analyses highlight an essential asymmetry between target control and source estimation.

1. Introduction

The observability and controllability of network dynamics has garnered considerable attention in the controls community in recent years (Liu, Slotine, & Barabási, 2013; Rahmani, Ji, Mesbahi, & Egerstedt, 2009). A primary focus of this effort has been to develop graph-theoretic sufficient or necessary conditions. More recently, metrics for the required control effort and the state estimation fidelity have also been characterized (Dhal & Roy, 2016; Pasqualetti, Zampieri, & Bullo, 2014), as a step toward sensor or actuator placement (Summers & Lygeros, 2014).

Many dynamical network applications only require guidance or estimation of a subset of the network's nodes. Recently, partial control of a network's dynamics has been studied under the heading of *target controllability* or reachability (Van Waarde, Kanat Camlibel, & Trentelman, 2017; Vosughi, Johnson, Roy, Warnick, & Xue, 2017). The purpose of this technical note is to examine metrics for the effort (energy) required for target control for a simple dynamical-network model, focusing on the base case that a single target node is being guided. A commensurate metric for source estimation, specifically for the error in estimating of a source node's state from noisy measurements, is also considered.

The main contributions of this note are to (1) develop spectral and graph-theoretic analyses of the target-control and source-estimation metrics, and (2) compare the metrics. The comparison demonstrates that source estimation is the harder problem, in the sense that metric value is larger. Further, the target-control metric exhibits a spatial pattern related to separating cutsets of the network's graph, while the source estimation metric does not.

Preliminary results in this direction were presented in Vosughi et al. (2017).

2. Problem formulation

A network with n nodes labeled $i = 1, \ldots, n$ is considered. Each node i has a scalar state $x_i[k]$ that evolves in discrete time. The network's full state $\mathbf{x}[k] = \begin{bmatrix} x_1[k] & \cdots & x_n[k] \end{bmatrix}^T$ is governed by a stable discrete-time linear dynamical model with state matrix A. Further, the dynamics are amenable to actuation at one node s, which we call the *source node*; and measurement at a second node t, which we call the *target node*. Formally, the dynamics are

$$\mathbf{x}[k+1] = A\mathbf{x}[k] + \mathbf{e}_s u[k], \qquad (1)$$
$$y[k] = \mathbf{e}_t^T \mathbf{x}[k] + N[k]$$

where we use the notation $\mathbf{e}_q$ for a 0–1 indicator vector with entry q equal to 1, and the scalar signal $u[k]$ is the input, $y[k]$ is the measurement, and $N[k]$ is a zero-mean unit-variance white Gaussian noise signal.

The first problem of interest is to characterize the effort required to move the state at the target node to a desired value by designing the input at the source node. The network is assumed to be initially relaxed ($\mathbf{x}[0] = \mathbf{0}$). The input $u[k]$ is to be designed

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to move the state at the target node to a unit value (without loss of generality) at time $\widehat{k}$, so that $x_t[\widehat{k}] = 1$. Our main interest is to characterize the minimum input energy, measured in a two-norm sense, required to achieve this goal. Formally, the target-control effort or energy over a horizon $\widehat{k}$ is defined as follows:

$$E(\widehat{k}) = min_{u[0],\ldots,u[\widehat{k}-1]} \sum_{k=0}^{\widehat{k}-1} u^2[k] \tag{2}$$

Subject to the goal state being achieved ($x_t[\widehat{k}] = 1$). The analysis will primarily focus on the minimum energy when ample time is available for control, i.e. $E = \lim_{\widehat{k}\to\infty} E(\widehat{k})$, which lower bounds the energy required over a finite horizon. It is important to stress that the formulation places no requirements on any network states except at the target node.

The second problem of interest is to characterize the fidelity with which the initial state at the source node can be estimated from the noisy measurements at the target node. The initial state $\mathbf{x}[0]$ is assumed to be an unknown, nonrandom vector, and a zero input $u[k]$ is assumed (the analysis is identical if the input is nonzero but known). Our interest is in determining the highest fidelity with which the initial state of the source node $x_s[0]$ can be estimated from the sequence of observations $y[0], \ldots, y[\widehat{k}-1]$. Specifically, the estimation error metric $F(\widehat{k})$ is defined as the minimum achievable mean-square error in the estimate among unbiased estimators of the source node's state. In analogy with the target control problem, the analysis will primarily focus on the metric value when ample data is available, i.e. $F = \lim_{\widehat{k}\to\infty} F(\widehat{k})$.

To permit topological analyses of the target-control effort and source-estimation error, a weighted digraph Γ is associated with the state matrix of the dynamical model. Γ is defined to have n vertices labeled $1, \ldots, n$, which correspond to the n network nodes. An edge is drawn from vertex i to vertex j in the graph (where i and j are not necessarily distinct) if A_{ji} is non-zero. The presence of the edge indicates that the next state of vertex j depends on the current state of vertex i. We note that in general the edge weights may be of either sign, however some of our analyses depend on the state matrix and hence edge weights being nonnegative; this is clarified in the presentation of results.

3. Target control effort

Algebraic conditions under which target control is possible, and expressions for the minimum energy required for target control, are established in the literature on output controllability (Kreindler & Sarachik, 1964). An input sequence can be designed to achieve target control over the horizon $\widehat{k}$, if and only if the impulse response $h[k] = \mathbf{e}_t^T A^k \mathbf{e}_s$ is non-zero for some $k = 0, \ldots, \widehat{k}-1$. We assume from here on that target control is possible. In this case, the minimum energy required for target control over the horizon $\widehat{k}$ is as follows:

$$E(\widehat{k}) = \frac{1}{\sum_{k=0}^{\widehat{k}-1} (\mathbf{e}_t^T A^k \mathbf{e}_s)^2} \tag{3}$$

The target-control energy is the inverse of sth diagonal entry of the controllability Gramian that can be calculated as follows:

$$G_c(0, \widehat{k}) = \sum_{k=0}^{\widehat{k}-1} A^k \mathbf{e}_s \mathbf{e}_s^T (A^T)^k \tag{4}$$

The minimum energy required for target control can readily be expressed in terms of the spectrum of the matrix A, provided that the system is asymptotically stable. For convenience, we present the result for the case that the eigenvalues of A are not defective. In this case, the infinite-horizon target control energy is given by

$$E = \frac{1}{\sum_{i=1}^{n} (\frac{v_{it} w_{is}}{1-\lambda_i})^2}, \tag{5}$$

where λ_i, $i = 1, \ldots, n$ are the n eigenvalues of A, and $\mathbf{v}_i$ and $\mathbf{w}_i$ are the right and left eigenvectors of A associated with λ_i (normalized to unit length), and we use the notation v_{iq} for the qth entry of $\mathbf{v}_i$ (respectively for $\mathbf{w}_i$). The expression follows immediately from substituting the eigenvalue decomposition of A (Rugh, 1996) into the expression for the target-control energy, hence details are omitted. The spectral expression indicates that target control is easy (requires little energy) if the state matrix has an eigenvalue close to 1 whose left eigenvector has a large entry corresponding to the source location, and whose right eigenvector has a large entry corresponding to the target location. In fact, the required control energy is limited if there is any eigenvalue λ_i such that $v_{it} w_{is}$ is adequately large, since the magnitude of $1 - \lambda_i$ is upper bounded by 2 for any stable system. This analysis shows that efficient target control only requires the ability to manipulate the target state via one controllable mode.

The algebraic and spectral expressions are a starting point for graph-theoretic analyses of the target-control energy, for classes of networks. The main graph-theoretic result developed here identifies a spatial pattern in the target control energy for different target locations, which is related to cutsets of the graph Γ. Prior to the presentation of this main result, several simple graphical results on the target-control task and energy are noted:

(1) Finite-energy target control is always possible for any horizon $\widehat{k} \geq 1$ if the source and target locations are the same, since $h[0] \neq 0$ in this case.

(2) Finite-energy target control over the horizon $\widehat{k}$ is possible only if the network graph Γ has a directed path from the source vertex to the target vertex of length less than or equal to $\widehat{k}$, since otherwise the impulse response is identically zero. In the case that the state matrix A is nonnegative or Metzler, the condition is necessary and sufficient. This is true because $\mathbf{e}_t^T A^k \mathbf{e}_s$ is necessarily positive in this case, where k is the distance between the source and target vertex in the network graph (Berman & Plemmons, 1994). Likewise, if the network's graph has a *unique* directed path of minimum length from the source vertex to the target vertex, then finite-energy target control is guaranteed even if the matrix A is not Metzler. We point the reader to Van Waarde et al. (2017) for graph-theoretic results on target controllability for nonnegative A, in the case where there are multiple targets.

(3) For a network with nonnegative state matrix A, if the weight of any edge in the graph Γ is increased or a new edge is added to graph, the minimum energy for target control $E(k)$ does not increase for any horizon k. This can be verified by noticing that powers of the state matrix are greater in an entry-wise sense when an edge weight is increased (Berman & Plemmons, 1994), and hence the energy decreases per Eq. (3). The energy strictly decreases for a sufficiently-long horizon, provided that there is a path from the source to the target which includes the modified/added edge.

(4) Consider a network with nonnegative state matrix A, and suppose that there is at least one directed path from the source vertex to the target vertex in the network graph. The notation $(s, q_1, q_2, \ldots, q_r, t)$ is used for the path whose product of edge weights $A_{q_1 s} A_{q_2 q_1} \ldots A_{t q_r}$ is largest among all paths between s and t. From properties of nonnegative matrices (Berman & Plemmons, 1994), it is immediate that the energy required for target control over the infinite horizon is upper bounded as follows:

$$E \leq \frac{1}{(A_{q_1 s} A_{q_2 q_1} \ldots A_{t q_r})^2} \tag{6}$$

The above results show that the target-control energy is small for any source and target pair, for dense network graphs.

The next main graphical result compares the target control energy for different possible target locations, for a class of diffusion-like network processes. The class of state matrices considered in this analysis is based on the following definition:

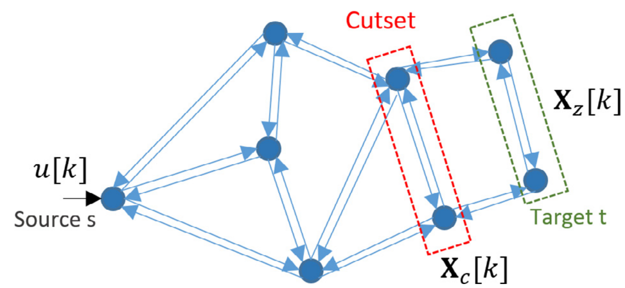

Fig. 1. Illustration of partitions by a node cutset.

Definition 1. The state matrix A is said to be an *α-diffusive matrix*, if it is a nonnegative matrix with row sums less than or equal to α, where $0 < \alpha \le 1$.

We notice that the class of α-diffusive matrices encompasses common models for network consensus, synchronization, and diffusion processes (e.g. Xiao and Boyd (2004)).

The result requires some further notation. To introduce this notation, let us first consider the target control energy $E(\widehat{k})$ for a specific source and target pair. Now consider a vertex cutset separating the source vertex from this target vertex on the graph Γ (and not including the target vertex). Without loss of generality, let $i = 1, \ldots, m$ be the corresponding labels of these cutset vertices. For each of these vertices $i = 1, \ldots, m$, the notation $E_i(\widehat{k})$ is used for the target control energy when the node i (which we call a cutset node) is instead the target node (see Fig. 1). Additionally, the notation $\mathbf{x}_c[k]$ is used for a vector containing the states of the cutset nodes (i.e. $\mathbf{x}_c[k] = \begin{bmatrix} x_1[k] & \cdots & x_m[k] \end{bmatrix}^T$). Also, $\mathbf{x}_z[k]$ is used for a vector containing the states of all the nodes whose corresponding vertices are separated from the source vertex by the cutset in Γ. Note that $\mathbf{x}_z[k]$ also contains the target state $x_t[k]$. Finally, we define the state matrix A to be α-diffusive (for some $\alpha \in [0, 1]$), if it is nonnegative and has row sums less than or equal to α.

The main result is a comparison of the target-control energy for cutset nodes vs the original target node:

Theorem 1. *Consider the target control task, in the case that the state matrix A is α-diffusive. Also consider a vertex cutset that separates the target from the source in the network graph Γ. Consider the original target control energy $E(\widehat{k})$, as well as the target control energies $E_i(\widehat{k})$ for target nodes $i = 1, \ldots, m$ on the cutset. Then $E_i(\widehat{k}) \le \alpha^2 E(\widehat{k})$ for some $i = 1, \ldots, m$.*

This comparison result depends on a key lemma:

Lemma 1. *Consider the target control task, in the case that the state matrix A is α-diffusive. Also consider a vertex cutset that separates the target from the source in the network graph Γ. For any input sequence at the source node that drives the target state to $x_t[\widehat{k}] = 1$, the state of at least one cutset node ($i = 1, \ldots, m$) will exceed $\frac{1}{\alpha}$ at some time before $\widehat{k}$. That is, $x_i[k] \ge \frac{1}{\alpha}$ for some $i = 1, \ldots, m$, and some $k = 0, \ldots, \widehat{k} - 1$.*

The lemma is proved first, followed by the theorem:

Proof (*Lemma*). The nodes are separated into three different groups: the cutset, the source partition containing the source node s, and the target partition containing the target node t. Since the target partition nodes only connect to the cutset nodes, the next-state vector for the target partition $\mathbf{x}_z[k+1]$ can be computed from the cutset state vector $\mathbf{x}_c[k]$ and the current target state vector. Specifically:

$$\mathbf{x}_z[k+1] = A_z \mathbf{x}_z[k] + B \mathbf{x}_c[k] \tag{7}$$

Where A_z is a principal submatrix of A, and $\begin{bmatrix} A_z & B \end{bmatrix}$ is also a submatrix (specifically a subset of the rows) of A. A_z has dimension $\hat{z} \times \hat{z}$, where $\hat{z}$ is the number of nodes in the target partition. Provided that the initial condition is zero, we then can write $\mathbf{x}_z[\widehat{k}]$ as

$$\mathbf{x}_z[\widehat{k}] = \sum_{k=0}^{\widehat{k}-1} A_z^k B \mathbf{x}_c[\widehat{k} - 1 - k] \tag{8}$$

Defining $P = \begin{bmatrix} B & A_z B & \cdots & A_z^{\widehat{k}-1} B \end{bmatrix}$, the above expression for $\mathbf{x}_z[\widehat{k}]$ can be rewritten as

$$\mathbf{x}_z[\widehat{k}] = P \begin{bmatrix} \mathbf{x}_c[\widehat{k} - 1] \\ \vdots \\ \mathbf{x}_c[0] \end{bmatrix} \tag{9}$$

Next, we show that the rows of matrix P sum to at most α. To do so, let us consider the matrix M as follows:

$$M \triangleq \begin{bmatrix} A_z & B \\ \mathbf{0} & I_m \end{bmatrix} \tag{10}$$

The first $\hat{z}$ rows of M have sums of at most α since $\begin{bmatrix} A_z & B \end{bmatrix}$ is a submatrix of A. It thus also follows that the first $\hat{z}$ rows of $M^{\widehat{k}}$ have sums of at most α. However,

$$M^{\widehat{k}} = \begin{bmatrix} A_z^{\widehat{k}} & \sum_{k=0}^{\widehat{k}-1} A_z^k B \\ \mathbf{0} & I_m \end{bmatrix} \tag{11}$$

Hence the $\sum_{k=0}^{\widehat{k}-1} A_z^k B$ has row sums of at most α. Noticing that the row sums of $\sum_{k=0}^{\widehat{k}-1} A_z^k B$ are identical to the row sums of P, we obtain that the rows in P have sums of at most α.

Finally, the target state satisfies

$$x_t[\widehat{k}] = P_i \begin{bmatrix} \mathbf{x}_c[\widehat{k} - 1] \\ \vdots \\ \mathbf{x}_c[0] \end{bmatrix} \tag{12}$$

Where P_i is a row of P, whose sum is at most α. Since $x_t[\widehat{k}] = 1$, at least one element in $\begin{bmatrix} \mathbf{x}_c[\widehat{k} - 1] \\ \vdots \\ \mathbf{x}_c[0] \end{bmatrix}$ must be at least $\frac{1}{\alpha}$. Thus, at least one cutset node's state is at least $\frac{1}{\alpha}$ for some $k = 0, 1, \ldots, \widehat{k} - 1$. ■

Proof (*Theorem*). Consider an optimal (minimal energy) input sequence $\tilde{u}[0], \ldots, \tilde{u}[\widehat{k} - 1]$, which drives $x_t[\widehat{k}]$ to 1. Since A is α-diffusive, it follows from Lemma 1 that at least one node on the node cutset reaches a value of at least $\frac{1}{\alpha}$ before $\widehat{k}$. That is, there is a cutset node $i \in \{1, \ldots, m\}$ such that $x_i[k_c] = h$ for some $h \ge \frac{1}{\alpha}$ and for some $k_c = 0, \ldots, \widehat{k} - 1$. Obviously, from causality, the subsequence $\tilde{u}[0], \ldots, \tilde{u}[k_c - 1]$ drives $x_i[k_c]$ to $h \ge \frac{1}{\alpha}$. Because the system (13) is linear and time-invariant, the following input sequence drives node i to 1 at time $\widehat{k}$:

$$u[0] = \cdots = u[\widehat{k} - k_c - 1] = 0, \tag{13}$$

$$u[\widehat{k} - k_c] = \frac{\tilde{u}[0]}{h}, \ldots, u[\widehat{k} - 1] = \frac{\tilde{u}[k_c - 1]}{h}$$

Thus, the minimum energy required to drive $x_i[\widehat{k}]$ to 1, i.e. $E_i(\widehat{k})$, is no larger than $\sum_{k=0}^{\widehat{k}-1} u^2[k] = \frac{1}{h^2} \sum_{k=0}^{k_c-1} \tilde{u}[k]$. Meanwhile, the minimum energy required to drive $x_t[\widehat{k}]$ to 1 is

$$E(\widehat{k}) = \sum_{k=0}^{\widehat{k}-1} \tilde{u}[k] \tag{14}$$

Hence, $E_i(\widehat{k}) \le \alpha^2 E(\widehat{k})$. ■

Theorem 1 demonstrates that, for α-diffusive systems, target control becomes increasingly difficult for graph cuts that are further away from the source vertex.

Remark 1. The graph-theoretic results developed in this section, including the preliminary conditions/bounds and the main cutset-based result (Theorem 1), depend on the state matrix having a diffusive structure, but do not require the state matrix to be non-defective. It is worth noting that the results can easily be extended to the case where the entries are positive and negative, but each absolute row sum is less than 1. Additionally, Theorem 1 can straightforwardly be extended to the multi-source case, in terms of cutsets separating all source vertices from the target. On the other hand, the spectral result (Eq. (5)) does not require diffusivity, but is developed for a non-defective state matrix; we note that the expression can be generalized to the case of a defective state matrix, however the presentation becomes more sophisticated.

Remark 2. Two directions of future work are worth noting. First, distributed computation of the target controllability metric may be valuable for on-line monitoring and decision-making. One possible approach for distributed computation may be to start from the spectral expression for the metric (Eq. (5)), and apply distributed algorithms for computing eigenvalues and eigenvector components (Kempe & McSherry, 2004). Second, noting the significant interest in time-varying diffusive processes (Blondel, Hendrickx, & Tsitsiklis, 2005), generalization of Theorem 1 to the time-varying case would be useful. We hypothesize that a similar result holds provided that a group of vertices can be found that persistently separate the source and target.

4. Source estimation error metric

In analogy with the target-control analysis, algebraic analyses of the source estimation error derive immediately from the standard treatment of observability using the Gramian (Kreindler & Sarachik, 1964). Finite-variance source estimation is possible if and only if the unobservable subspace of the pair $(\mathbf{e}_t^T, A)$ has no projection on $\mathbf{e}_s$. Further, provided that the pair is observable, the source estimation error over the horizon $\widehat{k}$ is as follows:

$$F(\widehat{k}) = \mathbf{e}_s^T \left(G_o(0,\widehat{k})\right)^{-1} \mathbf{e}_s \tag{15}$$

Where the observability Gramian $G_o(0,\widehat{k})$ calculates as follows:

$$G_o(0,\widehat{k}) = \sum_{k=0}^{\widehat{k}-1} (A^T)^k \mathbf{e}_t \mathbf{e}_t^T A^k \tag{16}$$

Similar expressions can also be developed in the atypical case that the pair $(\mathbf{e}_t^T, A)$ is not observable but source estimation is still possible, however details are omitted to save space.

The source estimation error over the infinite horizon can also be expressed in terms of the spectrum of A. This analysis is more intricate than for the target control energy, because the expression for $F(\widehat{k})$ involves the inverse of a Gramian. Let us here develop the expression, again in the case that A is not defective. For this case, the infinite-horizon observability Gramian $G_o \triangleq G_o(0,\infty) = \sum_{k=0}^{\infty}(A^T)^k \mathbf{e}_t \mathbf{e}_t^T A^k$ can be written as

$$G_o = (V^{-1})^T \begin{bmatrix} \frac{V_{1t}^2}{1-\lambda_1^2} & \frac{V_{1t}V_{2t}}{1-\lambda_1\lambda_2} & \cdots & \frac{V_{1t}V_{nt}}{1-\lambda_1\lambda_n} \\ \frac{V_{2t}V_{1t}}{1-\lambda_2\lambda_1} & \frac{V_{2t}^2}{1-\lambda_2^2} & \cdots & \frac{V_{2t}V_{nt}}{1-\lambda_2\lambda_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{V_{nt}V_{1t}}{1-\lambda_n\lambda_1} & \frac{V_{nt}V_{2t}}{1-\lambda_n\lambda_2} & \cdots & \frac{V_{nt}^2}{1-\lambda_n^2} \end{bmatrix} V^{-1} \tag{17}$$

The expression (17) has been developed by substituting the eigenvalue decomposition of A and then undertaking an algebraic simplification. The expression for the observability Gramian is closely related to a Cauchy matrix, which allows inversion using the standard Cauchy matrix inversion formula (see Rai et al. (2013)). In this way, an explicit expression for the source estimation error metric can be developed, as summarized in the following lemma:

Lemma 2. *The infinite-horizon source estimation error metric is given by*

$$F = \mathbf{e}_s^T V \begin{bmatrix} \frac{b_{11}\lambda_1}{V_{1t}^2} & \frac{b_{12}\lambda_2}{V_{2t}V_{1t}} & \cdots & \frac{b_{1n}\lambda_n}{V_{nt}V_{1t}} \\ \frac{b_{21}\lambda_1}{V_{1t}V_{2t}} & \frac{b_{22}\lambda_2}{V_{2t}^2} & \cdots & \frac{b_{2n}\lambda_n}{V_{nt}V_{2t}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{b_{n1}\lambda_1}{V_{1t}V_{nt}} & \frac{b_{n2}\lambda_2}{V_{2t}V_{nt}} & \cdots & \frac{b_{nn}\lambda_n}{V_{nt}^2} \end{bmatrix} V^T \mathbf{e}_s \tag{18}$$

or explicitly as

$$F = \sum_{j=1}^{n}\sum_{i=1}^{n} b_{ij}\lambda_j \frac{V_{is}V_{js}}{V_{it}V_{jt}}, \tag{19}$$

where $b_{ij} = -\frac{\prod_{k=1}^{n}(\frac{1}{\lambda_j}-\lambda_k)(\frac{1}{\lambda_k}-\lambda_i)}{(\frac{1}{\lambda_j}-\lambda_i)(\prod_{1\le k\le n, k\ne j}(\frac{1}{\lambda_j}-\frac{1}{\lambda_k}))(\prod_{1\le k\le n, k\ne i}(\lambda_i+\frac{1}{\lambda_k}))}$.

The spectral expression for the source estimation error has a very different form compared to that for the target control energy. The expression shows that source estimation is difficult – i.e., the error metric is large – whenever any one mode has a significant projection at the source node and a small projection at the target node; this is in sharp contrast with target control, where the controllability of any one mode makes the metric small. Also, source estimation is difficult for some source and target pairs if any two eigenvalues of the state matrix are close. Thus, source estimation is necessarily difficult for large networks with symmetric state matrices, since such state matrices have closely-placed eigenvalues.

Remark. The inversion of the Gramian, and hence the spectral computation of the source estimation error, becomes more sophisticated when the state matrix is defective. The inversion of the Gramian for a single Jordan block is addressed in Dhal, Lafferriere, and Caughman (2016).

The asymmetry between source estimation and target control is also clarified by an explicit comparison between the two metrics for a specified source and target pair (as in our problem formulation). The following lemma formalizes that source estimation is always as hard as target control, in the sense that the estimation error metric majorizes the target control metric:

Lemma 3. *For any network model, source and target location, and horizon $\widehat{k}$, the target-control energy metric is majorized by the source-estimation error metric, i.e. $E(\widehat{k}) \le F(\widehat{k})$.*

Proof. The target-control energy metric is given by

$$E(\widehat{k}) = \frac{1}{\sum_{k=0}^{\widehat{k}-1}(\mathbf{e}_t^T A^k \mathbf{e}_s)^2} \tag{20}$$

Noticing that $\mathbf{e}_t^T A^k \mathbf{e}_s$ is a scalar, the metric can be rewritten as

$$E(\widehat{k}) = \frac{1}{\sum_{k=0}^{\widehat{k}-1} \mathbf{e}_s^T (A^T)^k \mathbf{e}_t \mathbf{e}_t^T A^k \mathbf{e}_s} \tag{21}$$

Thus, $E(\widehat{k})$ is equal to the inverse of the sth diagonal entry of the observability Gramian. Meanwhile the source-estimation error metric $F(\widehat{k})$ is the sth diagonal entry of the inverse of the observability Gramian. Since the observability Gramian is a symmetric positive definite matrix, it follows immediately that the inverse of a diagonal entry is less than or equal to the corresponding diagonal entry in the inverse matrix. Thus, it follows that $E(\widehat{k}) \leq F(\widehat{k})$. ■

A simple example can be used to demonstrate that the source estimation error metric does not exhibit many of the simple topological patterns displayed by the target control metric. Specifically, consider a network with four nodes, with state matrix $A = \begin{bmatrix} 0.5 & 0.25 & 0.25 & 0 \\ .25 & 0 & 0 & 0.15 \\ .25 & 0 & 0 & 0.15 \\ 0 & 0.1 & 0.15 & 0 \end{bmatrix}$, where node 4 is the target. The infinite horizon source estimation errors for each possible source location are given by $F_1 = 250$, $F_2 = 3.6E4$, $F_3 = 1.7E4$, and $F_4 = 1.0$. Thus, we see that the source estimation error does not increase along partitions away from the target location: it is easier to estimate the state at Node 1 than at either Node 2 or Node 3. This is the case because the states of Nodes 2 and 3 are nearly indistinguishable, due to the almost-symmetric structure of the network. Also, it can easily be shown that finite-variance estimation becomes impossible when Node 2 is the source, if $A_{4,2}$ is increased to 0.15. Thus, increasing edge weights does not necessarily make source estimation easier.

The analyses of the source estimation error metric demonstrate an essential asymmetry between target control and source estimation. This asymmetry arises because low-energy target control only requires that one modal direction with a projection at the target location is easily controllable, while low-error source estimation requires that all modal directions with a projection at the source location are easily observable.

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