Necessary and Sufficient Conditions on State Transformations that Preserve the Causal Structure of LTI Dynamical Networks

Humphrey Leung and Sean Warnick†

***Abstract*— Linear time-invariant (LTI) dynamic networks are described by their dynamical structure function, and generally they have many possible state space realizations. This work characterizes the necessary and sufficient conditions on a state transformation that preserves the dynamical structure function, thereby generating the entire set of realizations of a given order for a specific dynamic network.**

I. Introduction

Advancement of technology and the need for modules with highly sophisticated functions have led to the emergence of complex systems despite of the raising price for system maintenance. To better manage complicated systems and procedures, people across different industries have been putting effort into building systems with particular structures, for example, systems with highly cohesive and loosely coupled modules are often preferable in computer programming. This work is a step toward devising guiding principles for designing structured systems such as distributed controllers by characterizing equivalent classes of systems with the same structure. Specifically, we present the necessary and sufficient conditions on a state transformation such that the system's dynamical structure function is preserved will be presented in this work as a way to facilitate designs of distributed controllers and structured systems.

This work is a follow up inquiry of network semantics which was started in [1]. The study of dynamical systems has flourished in the past decades in relation to the development of related mathematical techniques and the recent growth of public accessible computation power. Different techniques for predicting and controlling dynamical systems has also arised as an answer to the needs of interpreting and stabilizing the growing complexity of artificial systems. Dynamical structure function is a mathematical model for capturing structural information of a dynamical system. It is also the theoretical foundation of network reconstruction, which is a technique for finding out inferences between signals and representing them as matrices of rational polynomials.

[1] is a work on proving the well-definedness of dynamical structure function, it guarantees a sensible meaning of a DSF by exploring the set of state space representations which it characterizes. A natural follow up question is that which of these state space representations with the same dynamical structure function are in common with in terms of some state transformations. The goal of this work is to answer this question by characterizing equivalent classes of state space realizations with respect to state transformations and dynamical structure function.

Humphrey Leung (moc.liamgb@gmail.com) and Sean Warnick (sean.warnick@gmail.com) are with the Information and Decision Algorithms Laboratories, Department of Computer Science, Brigham Young University, Provo, UT 84602, USA.

In the rest of this introduction section, we will go through the derivation of dynamical structure function and introduce a formal statement of our problem. Then in section 2, besides going deeper into other related literatures of modeling dynamical systems and theory, we will explore reasons why state space representations can be partitioned into equivalent classes with respect to the transfer function or dynamical structure function characterizing them, after that, a sufficient condition on a state transformation for preserving the network structure will be given to illustrate the direction of the main result. In the main result section, the problem is broken into four parts,

1) **(Structural Canonical Form of State Space Models)**, which allow us to work with a more handleable subset of state transformation without loss of generality,
2) **(Structural Parameters of a System)**, which establish a bidirectional relation between an simplified version of the problem and our final goal,
3) **(Necessary and Sufficient Condition for Preserving Dynamical Information Function)**, this section introduces techniques for traversing the set of state space representations with the same dynamical information function,
4) **(Necessary and Sufficient Condition for Preserving Dynamical Structure Function)**, this section introduces techniques for traversing the set of dynamical information functions with the same dynamical structure function.

By the end, we will come back to the definition of dynamical structure function and justify some important assumptions we have made.

A. Dynamical Structure Function

Consider a state space representation of a system

$$\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx + Du.\end{aligned}$$

The signal structure of a linear time invariant system is characterized by the following equation [1],

$$Y = QY + PU, \tag{1}$$

where the derivation of (Q, P) from (A, B, C, D) are as follow. We first partition (A, B, C, D) into block matrices commensurate with the partitioning of manifest variables Y

and hidden variables X. We assume that $C = \begin{bmatrix} I & 0 \end{bmatrix}$ and $D = 0$, the justification of such assumption has been given in [1] and will be restated again in the last section of this paper,

$$
\begin{aligned}
s\begin{bmatrix} Y \\ X \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} Y \\ X \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U \\
Y &= \begin{bmatrix} I & 0 \end{bmatrix}\begin{bmatrix} Y \\ X \end{bmatrix}.
\end{aligned} \tag{2}
$$

By expanding equation (2), we obtain

$$sY = A_{11}Y + A_{12}X + B_1U \tag{3}$$
$$sX = A_{21}Y + A_{22}X + B_2U. \tag{4}$$

Now, we rearrange equation (4) to be

$$X = (sI - A_{22})^{-1}(A_{21}Y + B_2U), \tag{5}$$

and by substituting equation (5) into equation (3), we have

$$sY = A_{11}Y + A_{12}(sI - A_{22})^{-1}(A_{21}Y + B_2U) + B_1 \tag{6}$$
$$
\begin{aligned}
sY = {} & (A_{11} + A_{12}(sI - A_{22})^{-1}A_{21})Y \\
& + (A_{12}(sI - A_{22})^{-1}B_2 + B_1)U.
\end{aligned} \tag{7}
$$

Definition 1.1 (Dynamical Information Function): Now, we can define $W(s)$ and $V(s)$ as the following,

$$W(s) = A_{12}(sI - A_{22})^{-1}A_{21} + A_{11}, \tag{8}$$
$$V(s) = A_{12}(sI - A_{22})^{-1}B_2 + B_1. \tag{9}$$

And the tuple (W, V) is called the *dynamical information function(DIF)* of the system.

In this paper, we call (W, V) the *dynamical information function* of the system because it contains all causal linkages of a system including its self-influencing edges. Now, equation (7) can be rewritten as

$$sY = W(s)Y + V(s)U. \tag{10}$$

For computational convenience, we further manipulate $W(s)$ and $V(s)$ by subtracting the diagonal of $W(s)$ on both side and move s to the right,

$$(sI - diag(W(s)))Y = (W(s) - diag(W(s)))Y + V(s)U, \tag{11}$$

then by moving $(sI - diag(W(s)))$ to the right, we get

$$
\begin{aligned}
Y = {} & (sI - diag(W(s)))^{-1}(W(s) - diag(W(s)))Y \\
& + (sI - diag(W(s)))^{-1}V(s)U.
\end{aligned} \tag{12}
$$

Definition 1.2 (Dynamical Structure Function): With equation (12), we define (Q, P) as

$$Q(s) = (sI - diag(W(s)))^{-1}(W(s) - diag(W(s))), \tag{13}$$
$$P(s) = (sI - diag(W(s)))^{-1}V(s). \tag{14}$$

The tuple (Q, P) is called as the *dynamical structure function(DSF)* of the system.

In contrast with the *dynamical information function* (W, V), the *dynamical structure function* (Q, P) contains only the structure of our interests, that is, every non self-influencing dynamics of the system.

Furthermore, the relationship between transfer function and DSF is as follow,

$$
\begin{aligned}
Y &= QY + PU && (15)\\
&= (I - Q)^{-1}PU && (16)\\
&= G(s)U. && (17)
\end{aligned}
$$

Thus, G can be expressed as $G(s) = (I - Q)^{-1}P$. Since the number of variables in the (Q, P) tuple is greater than that of $G(s)$, (Q, P) captures more information about the system, and we regard those rational polynomials in (Q, P) as *structural dynamics* between manifest and input variables. Furthermore, Q is a rational polynomial matrix capturing casual/feedback dynamics between manifest variables, and P is a rational polynomial matrix capturing casual dynamics between manifest variables and inputs.

B. Problem Statement

With the definition of *dynamical structure function* given above, we can now formulate our problem statement as follow:

Given a system

$$
\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx + Du,
\end{aligned}
$$

with a dynamical structure function (Q, P), what is the necessary and sufficient conditions on a state transformation T such that the new system

$$
\begin{aligned}
\dot{x} &= TAT^{-1}x + TBu, \\
y &= CT^{-1}x + Du
\end{aligned}
$$

has the same dynamical structure function (Q, P) as the original system.

II. Related Works, Background and Motivation

A. Related Work

There are couple concurrent models[2, 3, 4] attempting to capture causal relationships between system variables. Some of them are proper subset of DSF, for example, linear dynamical graph suggested by Materassi and Innocenti replaced P in (Q, P) by I and assume all input to be i.i.d random noises. On the other hand, there are models of which DSF is a proper subset of, and there are also models that are seemingly unrelated to the theoretical basis of DSF, for example, the Bayesian Graphical Models[5]. But all of these models leverage dependency graph to convey the structure of causal network within a system.

Chetty and Warnick in [6]classify system representations with respect to the computation granularity of the computation model, from the most abstract manifest structures to the most detailed complete computational structures. In a following paper, [1], efforts were made to justify the definition of dynamical structure functions. The techniques and insights of this work has opened up questions of partitioning state space models with respect to their dynamical structure function.

B. Partitioning of State Space Models

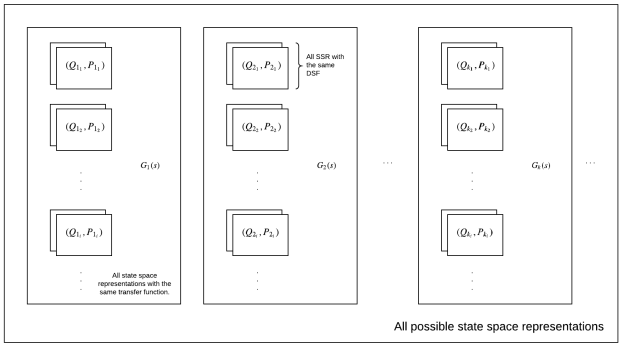

Fig. 1. Partitioning of state space representations(SSR) by their dynamical structure function(DSF) and transfer function(TF).

This section will focus on different system representations, for example, transfer function(TF), dynamical structure function(DSF) and state space representation(SSR), and how they are related. One of the goal in this section is to introduce the idea of partitioning state space representations into equivalent classes with the same transfer function or dynamical structure function. While transfer functions capture only the input-output mapping of a dynamical system, dynamical structure functions contain also dynamics between manifest variables. Moreover, state space models includes computation details of both hidden and manifest variables given inputs and initial state.

Proposition 2.1 (Existence and Uniqueness of DSF): Given any state space representation (A, B, C, D), there always exists an unique dynamical structure function (Q, P).

Proposition 2.2 (Existence and Uniqueness of TF): Given any dynamical structure function (Q, P), there always exists an unique transfer function G.

Since each state space representation has only one transfer function, but each TF can be realized by multiple state space representations, the set of state space representations can be partitioned by their corresponding transfer functions, and this relationship is true between TF and DSF, DSF and SSR also. Furthermore, since each DSF can only be characterized by one TF, thus each partition of SSR under DSF will not overlap with multiple partitions of state space representation under TF.

C. Sufficient Conditions on State Transformation T that Preserve (W, V)

Before diving into a discussion of sufficient conditions on state transformation T that preserve (W, V), we first define a new term *structurally equivalent* for convenient.

Definition 2.1 (Structural Equivalence): Two systems are *structurally equivalent* if they are characterized by the same dynamical information function (W, V).

We are interested in the sufficient conditions which would imply such equivalence, much like algebraic equivalence implies zero-state equivalence. A formal definition of algebraically equivalent can be found in [7],

Definition 2.2: Two continuous-time or discrete-time LTI systems

$$\begin{cases} \dot{x}/x^+ = Ax + Bu \\ y = Cx + Du \end{cases} \quad or \quad \begin{cases} \dot{\bar{x}}/x^+ = \bar{A}\bar{x} + \bar{B}u \\ y = \bar{C}\bar{x} + \bar{D}u, \end{cases}$$

respectively, are *algebraically equivalent* if there exists a nonsingular matrix T such that

$$\bar{A} := TAT^{-1}, \qquad \bar{B} := TB, \qquad \bar{C} := CT^{-1}, \qquad \bar{D} := D.$$

The corresponding map $\bar{x} = Tx$ is called a *similarity transformation* or an *equivalent transformation*.

Intuitively, a sufficient condition on T for preserving (W, V) should be similar to but more restrictive than algebraic equivalence, since such condition would partition a class of algebraically equivalent state space representations into subclasses of Structurally equivalent SSRs which are characterized by the same (W, V).

Theorem 2.3 (Sufficient Conditions for Preserving DIF): Restricting T to be the following

$$T = \begin{bmatrix} I & 0 \\ 0 & T_4 \end{bmatrix},$$

is a sufficient condition of preserving dynamical information function of a state space representation on the state transformation T.

Proof: The proof can be broken into two parts, first by showing that $\bar{W} = W$, then we will show that $\bar{V} = V$. Before start, let's remind ourselves of the definition of DIF

$$\begin{aligned} W(s) &= A_{11} + A_{12}(sI - A_{22})^{-1}A_{21}, \\ V(s) &= A_{12}(sI - A_{22})^{-1}B_2 + B_1. \end{aligned}$$

Part 1: To show that $W(s) = \bar{W}(s)$, we can derive $(\bar{A}_{11}, \bar{A}_{12}, \bar{A}_{21}, \bar{A}_{22})$ with the given T.

$$\begin{aligned} TAT^{-1} &= \begin{bmatrix} I & 0 \\ 0 & T_4 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & T_4^{-1} \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & T_4 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12}T_4^{-1} \\ A_{21} & A_{22}T_4^{-1} \end{bmatrix} \\ &= \begin{bmatrix} A_{11} & A_{12}T_4^{-1} \\ T_4A_{21} & T_4A_{22}T_4^{-1} \end{bmatrix} \\ &= \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}. \end{aligned}$$

Therefore,

$$\begin{aligned} \bar{W}(s) &= \bar{A}_{11} + \bar{A}_{12}(sI - \bar{A}_{22})^{-1}\bar{A}_{21} \\ &= A_{11} + A_{12}T_4^{-1}(sI - T_4A_{22}T_4^{-1})^{-1}T_4A_{21} \\ &= A_{11} + A_{12}T_4^{-1}(T_4(sI - A_{22})T_4^{-1})^{-1}T_4A_{21} \\ &= A_{11} + A_{12}(sI - A_{22})^{-1}A_{21} \\ &= W(s) \end{aligned}$$

Part 2: To show that $V(s) = \bar{V}(s)$, we derive $(\bar{B}_1, \bar{B}_2)$ with the given T.

$$\begin{aligned} TB &= \begin{bmatrix} I & 0 \\ 0 & T_4 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\ &= \begin{bmatrix} B_1 \\ T_4B_2 \end{bmatrix}. \end{aligned}$$

Therefore,

$$\begin{aligned}\bar{V}(s) &= \bar{A}_{12}(sI-\bar{A}_{22})^{-1}\bar{B}_2+\bar{B}_1\\ &= A_{12}T_4^{-1}(sI-T_4A_{22}T_4^{-1})^{-1}T_4B_2+B_1\\ &= A_{12}T_4^{-1}(T_4(sI-A_{22})T_4^{-1})^{-1}T_4B_2+B_1\\ &= A_{12}(sI-A_{22})^{-1}B_2+B_1\\ &= V(s).\end{aligned}$$

■

D. Structural Canonical Form

In [1], Chetty and Warnick are interested in showing that dynamical structure function is well-defined for all state space representations, more specifically, they have shown that dynamical structure function as they have defined are invariant under

1) basis of null space of C and E, which justify the use of $C=[I \quad 0]$, and
2) state permutations, that is, any change of basis to the state variables x of a SSR.

These two conditions together guarantee DSF to be a well-defined notion of computation models which carry structural information of dynamic networks. We will demonstrate a formal definition of DSF by considering systems in the following form

$$\begin{aligned}\dot{x} &= Ax+Bu\\ y &= Cx+Du,\end{aligned}$$

where C is a fat matrix with full row rank. This is a rather restricted subset of SSRs compared to the set of SSRs being considered in [1], but since a full justification has been presented in previous papers, we will present a simplified version of the proof to point out some important intuitions.

Let a state transformation $T\in\mathbb{R}^{n\times n}$ be

$$T=\begin{bmatrix}C\\ E^T\end{bmatrix},$$

where E is a unitary matrix of any basis of $ker(C)$. And now, T^{-1} can be derived from the definition of T

$$T^{-1}=\begin{bmatrix}C^T(CC^T)^{-1} & E\end{bmatrix}.$$

Then we can apply a change of basis on the system by T, let $z=Tx$, where

$$\bar{C}=CT^{-1}=[CC^T(CC^T)^{-1} \quad CE]=[I \quad 0]$$

, and the new system is defined as:

$$\begin{aligned}\begin{bmatrix}\dot{z_1}\\ \dot{z_2}\end{bmatrix} &= \begin{bmatrix}\bar{A}_{11} & \bar{A}_{12}\\ \bar{A}_{21} & \bar{A}_{22}\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix}+\begin{bmatrix}\bar{B}_1\\ \bar{B}_2\end{bmatrix}u\\ y &= \begin{bmatrix}I & 0\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix}+\bar{D}u.\end{aligned} \tag{18}$$

Therefore, there always exist a state transformation such that an arbitrary state space representation (A,B,C,D) is algebraically equivalent to $(\bar{A},\bar{B},I,0)$.

E. Similarity Transform of Structural Canonical Form

It can be shown that if we restrict the subset of state space representations of interest to $(A,B,\begin{bmatrix}I & 0\end{bmatrix},0)$, then a state transformation T between two similar system in the set is forced to be a certain form, that is

$$T=\begin{bmatrix}I & 0\\ T_3 & T_4\end{bmatrix}.$$

Lemma 2.4: Suppose two systems $(A,B,\begin{bmatrix}I & 0\end{bmatrix},0)$ with state variables x and $(\bar{A},\bar{B},\begin{bmatrix}I & 0\end{bmatrix},0)$ with state variable z are algebraically equivalent, then there exists an invertible state transformation $T=\begin{bmatrix}T_1 & T_2\\ T_3 & T_4\end{bmatrix}$ where $T_1=I$, $T_2=0$ such that $z=Tx$.

Proof: Suppose $C=\bar{C}=\begin{bmatrix}I & 0\end{bmatrix}$, and let $T=\begin{bmatrix}T_1 & T_2\\ T_3 & T_4\end{bmatrix}$. Then by evaluating $\bar{C}$,

$$\begin{aligned}\bar{C} &= C\begin{bmatrix}T_1 & T_2\\ T_3 & T_4\end{bmatrix}\\ &= \begin{bmatrix}I & 0\end{bmatrix}\begin{bmatrix}T_1 & T_2\\ T_3 & T_4\end{bmatrix}\\ &= \begin{bmatrix}T_1 & T_2\end{bmatrix}\\ &= \begin{bmatrix}I & 0\end{bmatrix} \qquad \text{(since } \bar{C}=\begin{bmatrix}I & 0\end{bmatrix} \text{ in the restricted subset of state space representations).}\end{aligned}$$

■

We are going to work with this restricted subset of state space representations until section E.

III. Main Results

A. Structural Parameters of a System

We borrow the notion of Markov parameters and extends it to an analogous notion called *structural parameters* for dynamical structure function in this section. The following is true from the definition of Laplace transform and matrix exponential:

$$(sI-A)^{-1}=\mathcal{L}[e^{At}]=:\mathcal{L}[\sum_{i=0}^{\infty}\frac{t^i}{i!}A^i],$$

and

$$(sI-A)^{-1}=\sum_{i=0}^{\infty}s^{-(i+1)}A^i$$

through replacing $\mathcal{L}[\frac{t^i}{i!}]$ by $s^{-(i+1)}$. Hence, W and V can be rewritten in the following form,

$$W=A_{11}+\sum_{i=0}^{\infty}s^{-(i+1)}A_{12}A_{22}^iA_{21}, \tag{19}$$

$$V=B_1+\sum_{i=0}^{\infty}s^{-(i+1)}A_{12}A_{22}^iB_2. \tag{20}$$

Furthermore, the i^{th} derivative of the impulse response of W and V when $t\to 0$ can be written as $A_{12}A_{22}^iA_{21}$ and $A_{12}A_{22}^iB_2$ $\forall i\geq 1$ respectively.

Definition 3.1 (Structural Parameters): The structural parameters of a state space representation of a system are

$$\begin{aligned} &A_{11}, \qquad A_{12}A_{22}^iA_{21}, \\ &B_1, \qquad A_{12}A_{22}^iB_2, \qquad \forall i \geq 0. \end{aligned}$$

Lemma 3.1 (Structural Parameters Preserve DIF): Two state space representations

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \quad or \quad \begin{cases} \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u \\ y = \bar{C}\bar{x} + \bar{D}u, \end{cases}$$

realize the same *dynamical information function* (W, V) if and only if they have the same *structural parameters*, that is,

$$\begin{aligned} A_{11} &= \bar{A}_{11}, \qquad A_{12}A_{22}^iA_{21} = \bar{A}_{12}\bar{A}_{22}^i\bar{A}_{21}, \\ B_1 &= \bar{B}_1, \qquad A_{12}A_{22}^iB_2 = \bar{A}_{12}\bar{A}_{22}^i\bar{B}_2. \end{aligned}$$

Proof: From equation (19) and (20), we know that two systems would have the same dynamical information function if they have the same structural parameters.

On the other hand, if two systems have the same dynamical information function, that is $W = \bar{W}$ and $V = \bar{V}$, then they must have the same A_{11} and B_1. This can be shown by setting s to infinity. Furthermore, since W and V are matrices of transfer functions, and equivalence of transfer functions implies equivalence of their impulse responses, we can conclude that the two systems have the same $A_{12}A_{22}^iA_{21}$ and $A_{12}A_{22}^iB_2$ $\forall i$. ■

B. The Necessary and Sufficient Condition on Transformation T that Preserve (W, V)

Theorem 3.2: Two restricted LTI systems

$$\begin{cases} \dot{x} = Ax + Bu \\ y = [I \quad 0]x \end{cases} \quad or \quad \begin{cases} \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u \\ y = [I \quad 0]\bar{x}, \end{cases}$$

respectively, are *structurally equivalent* if and only if there exists a nonsingular matrix $T = \begin{bmatrix} I & 0 \\ T_3 & T_4 \end{bmatrix}$ such that

$$A_{12}A_{22}^iT_4^{-1}T_3 = 0 \qquad \forall i \geq 0.$$

Proof: From the previous section, we have shown that two systems are structurally equivalent if and only if they carry the same structural parameters. We are going to show that the two systems carry the same structural parameters if and only if there exists an invertible transformation $T = \begin{bmatrix} I & 0 \\ T_3 & T_4 \end{bmatrix}$ such that

$$A_{12}A_{22}^iT_4^{-1}T_3 = 0 \qquad \forall i \geq 0.$$

Let's define a $\bar{A}$ and $\bar{B}$ with a non-singular transformation T such that

$$\begin{aligned} \bar{A} &= TAT^{-1} \\ &= \begin{bmatrix} I & 0 \\ T_3 & T_4 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ -T_4^{-1}T_3 & T_4^{-1} \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ T_3 & T_4 \end{bmatrix} \begin{bmatrix} A_{11} - A_{12}T_4^{-1}T_3 & A_{12}T_4^{-1} \\ A_{21} - A_{22}T_4^{-1}T_3 & A_{22}T_4^{-1} \end{bmatrix} \\ &= \begin{bmatrix} A_{11} - A_{12}T_4^{-1}T_3 & A_{12}T_4^{-1} \\ \begin{matrix} T_3A_{11} + T_4A_{21} \\ -(T_3A_{12} + T_4A_{22})T_4^{-1}T_3 \end{matrix} & (T_3A_{12} + T_4A_{22})T_4^{-1} \end{bmatrix} \\ &= \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}. \end{aligned}$$

Also:

$$\begin{aligned} \bar{B} &= TB \\ &= \begin{bmatrix} I & 0 \\ T_3 & T_4 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\ &= \begin{bmatrix} B_1 \\ T_3B_1 + T_4B_2 \end{bmatrix} \\ &= \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} \end{aligned}$$

Part 1: ($\Rightarrow$). Suppose the two systems carry the same structural parameters, that is,

$$\begin{aligned} A_{11} &= \bar{A}_{11}, \qquad A_{12}A_{22}^iA_{21} = \bar{A}_{12}\bar{A}_{22}^i\bar{A}_{21}, \\ B_1 &= \bar{B}_1, \qquad A_{12}A_{22}^iB_2 = \bar{A}_{12}\bar{A}_{22}^i\bar{B}_2. \end{aligned}$$

Then

$$\begin{aligned} A_{11} &= \bar{A}_{11} \\ A_{11} &= A_{11} - A_{12}T_4^{-1}T_3 \\ A_{12}T_4^{-1}T_3 &= 0. \end{aligned}$$

Therefore,

$$A_{12}A_{22}^iT_4^{-1}T_3 = 0$$

when $i = 0$.

Assume $A_{12}A_{22}^iT_4^{-1}T_3 = 0$ for all $0 \leq i \leq k$ for some $k \geq 0$. Then,

$$\begin{aligned} A_{12}A_{22}^kA_{21} &= \bar{A}_{12}\bar{A}_{22}^k\bar{A}_{21} \\ &= (A_{12}T_4^{-1}) \\ &\quad ((T_3A_{12} + T_4A_{22})T_4^{-1})^k \\ &\quad (T_3A_{11} + T_4A_{21} - \\ &\quad (T_3A_{12} + T_4A_{22})T_4^{-1}T_3) \\ &= A_{12}A_{22}^kT_4^{-1}(T_3A_{11} + T_4A_{21} - \\ &\quad (T_3A_{12} + T_4A_{22})T_4^{-1}T_3) \\ &= A_{12}A_{22}^kT_4^{-1}(T_4A_{21} \\ &\quad - T_4A_{22}T_4^{-1}T_3) \\ A_{12}A_{22}^{k+1}T_4^{-1}T_3 &= 0. \end{aligned}$$

Notice that $(A_{12}T_4^{-1})((T_3A_{12} + T_4A_{22})T_4^{-1})^k$ is evaluated to $A_{12}A_{22}^kT_4^{-1}$ because any terms in the expansion of $(A_{12}T_4^{-1})((T_3A_{12} + T_4A_{22})T_4^{-1})^k$

composed with at least one $T_3 A_{12} T_4^{-1}$ will have the form $A_{12} A_{22}^l T_4^{-1} T_3 A_{12} \Phi$ where $0 \leq l < k$, and are evaluated to 0 by the inductive assumption.

Therefore, by principle of mathematical induction,

$$A_{12} A_{22}^i T_4^{-1} T_3 = 0 \qquad \forall i \geq 0$$

if the two systems carry the same structural parameters.

Part 2: ($\Leftarrow$). By definition of $\bar{B}$, we have $B_1 = \bar{B}_1$. Suppose:

$$A_{12} A_{22}^i T_4^{-1} T_3 = 0 \qquad \forall i \geq 0,$$

then by definition of $\bar{A}_{11}$, $\bar{A}_{11} = A_{11} - A_{12} T_4^{-1} T_3 = 0$, also:

$$\begin{aligned}
\bar{A}_{12} \bar{A}_{22}^i \bar{A}_{21} &= (A_{12} T_4^{-1})((T_3 A_{12} + T_4 A_{22}) T_4^{-1})^i \\
&\quad (T_3 A_{11} + T_4 A_{21} - \\
&\quad (T_3 A_{12} + T_4 A_{22}) T_4^{-1} T_3) \\
&= A_{12} A_{22}^i T_4^{-1} (T_3 A_{11} + T_4 A_{21} - \\
&\quad (T_3 A_{12} + T_4 A_{22}) T_4^{-1} T_3) \\
&= A_{12} A_{22}^i T_4^{-1} (T_4 A_{21} - T_4 A_{22} T_4^{-1} T_3) \\
&= A_{12} A_{22}^i A_{21}.
\end{aligned}$$

The same argument applies to prove $\bar{A}_{12} \bar{A}_{22}^i \bar{B}_2 = A_{12} A_{22}^i B_2$ by replacing A_{21} with B_2 and $\bar{A}_{21}$ with $\bar{B}_2$.

Therefore, the two systems carry the same structural parameters, that is,

$$\begin{aligned}
A_{11} &= \bar{A}_{11}, \qquad A_{12} A_{22}^i A_{21} = \bar{A}_{12} \bar{A}_{22}^i \bar{A}_{21}, \\
B_1 &= \bar{B}_1, \qquad A_{12} A_{22}^i B_2 = \bar{A}_{12} \bar{A}_{22}^i \bar{B}_2.
\end{aligned}$$

With lemma (3.1), we have shown that (structural equivalence) $\Leftrightarrow$ (same structural parameters) $\Leftrightarrow$ ($A_{12} A_{22}^i T_4^{-1} T_3 = 0$). Therefore, two systems are structurally equivalent if and only if

$$A_{12} A_{22}^i T_4^{-1} T_3 = 0 \qquad \forall i \geq 0.$$

■

C. Equivalent classes of (W, V) characterized by the same (Q, P)

In this section, we will cover the condition for constructing equivalent classes of dynamical information functions which are characterized by the same dynamical structure function (Q, P).

Theorem 3.3: Two dynamical information functions (W, V) and $(\bar{W}, \bar{V})$ realize the same dynamical structure function (Q, P) if and only if there exist a transformation L such that

$$LV = \bar{V}, \qquad L(W - diag(W)) = \bar{W} - diag(\bar{W}).$$

Furthermore, $L = (sI - diag(\bar{W}(s)))(sI - diag(W(s)))^{-1}$.

Proof: By definition (1.2), (Q, P) are defined as

$$\begin{aligned}
Q(s) &= (sI - diag(W(s)))^{-1} (W(s) - diag(W(s))), \\
P(s) &= (sI - diag(W(s)))^{-1} V(s).
\end{aligned}$$

Let $D = (sI - diag(W(s)))$ and $\bar{D} = (sI - diag(\bar{W}(s)))$. (W, V) and $(\bar{W}, \bar{V})$ realize the same dynamical structure function (Q, P) if and only if they derive the same (Q, P), that is

$$\begin{aligned}
D^{-1}(W(s) - diag(W(s))) &= \bar{D}^{-1}(\bar{W}(s) - diag(\bar{W}(s))) \\
D^{-1} V(s) &= \bar{D}^{-1} \bar{V}(s).
\end{aligned}$$

Therefore,

$$\begin{aligned}
(\bar{W}(s) - diag(\bar{W}(s))) &= \bar{D} D^{-1} (W(s) - diag(W(s))) \\
\bar{V}(s) &= \bar{D} D^{-1} V(s).
\end{aligned}$$

Set $L := \bar{D} D^{-1}$, and it's clear that it is equivalent to the existence of a transformation L such that

$$\begin{aligned}
(\bar{W}(s) - diag(\bar{W}(s))) &= L(W(s) - diag(W(s))) \\
\bar{V}(s) &= LV(s).
\end{aligned}$$

■

IV. Conclusion

In this work, we have found a necessary and sufficient condition on a state transformation T such that the dynamical information function (W, V) might preserve. And it turns out that this condition has to do with the orthogonality between hidden dynamics of the system, that is A_{12} and A_{22}, and the reduced components of the state transformation, T_4^{-1} and T_3. This finding will allow us to answer questions related to information cost of realizing dynamical structure function in comparison with realizing state space model from transfer function, because of the added insight about the number of parameters which are invariant within an equivalent class of SSRs with the same DSF.

References

[1] V. Chetty and S. Warnick, "Network semantics of dynamical systems," in *Conference on Decision and Control*, Osaka, Japan, 2015.

[2] C. J. Quinn, N. Kiyavash, and T. P. Coleman, "Directed information graphs," *IEEE Transactions on Information Theory*, vol. 61, no. 12, pp. 6887–6909, Dec 2015.

[3] D. Materassi and G. Innocenti, "Topological identification in networks of dynamical systems," *IEEE Transactions on Automatic Control*, vol. 55, no. 8, pp. 1860–1871, Aug 2010.

[4] B. Bamieh and L. Giarre, "Identification of linear parameter varying models," in *Proceedings of the 38th IEEE Conference on Decision and Control (Cat. No.99CH36304)*, vol. 2, 1999, pp. 1505–1510 vol.2.

[5] J. Pearl, *Causality: Models, Reasoning and Inference*, 2nd ed. New York, NY, USA: Cambridge University Press, 2009.

[6] V. Chetty and S. Warnick, "Meanings and applications of structure in networks of dynamic systems," *CoRR*, vol. abs/1406.1844, 2014. [Online]. Available: http://arxiv.org/abs/1406.1844

[7] J. P. Hespanha, "Linear systems theory." Princeton university press, 2009.

[8] F. N. Hanne Riis Nielson, *Semantics with applications a formal introduction*. John Wiley & Sons, July 1999.