Dynamic Networks: Representations, Abstractions, and Well-Posedness

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***Abstract*— This paper introduces notions of abstraction and realization for dynamic networks. These processes generate dynamically equivalent representations of a system, but with varying degrees of structural detail. Nuanced definitions and associated conditions for maintaining well-posedness for these multi-resolution representations of a dynamic system are then detailed, ensuring that hierarchies of network representations are sensible as multi-resolution models of dynamic networks. Although the ideas are developed precisely here for LTI networks, many of the concepts remain fundamental in the nonlinear setting.**

I. Introduction

Dynamic networks have increasingly become the system representation of choice for a variety of applications where the modeling of both dynamic behavior and distributed structure is essential [1], [2]. This paper contributes to the theory of dynamic networks by introducing precise notions of abstractions and realizations of networks, as well as characterizing nuanced notions of well-posedness necessary to ensure that such multi-resolution descriptions of a system remain sensible.

A. Background: Dynamic Networks

Dynamic networks are representations of dynamic systems that are characterized by dynamic dependencies among manifest variables. We can often associate these dependencies with a (hyper)graph structure, with nodes identified as manifest variables and (hyper)edges defining dependencies among subsets of these variables. These dependencies lead to a notion of *signal structure*, characterized by the directed (hyper)graph representation among manifest variables of the dynamic system. The signal structures of linear systems are always (non-hyper) graphs, since pairwise dependencies on variables are always sufficient to describe the full dynamic behavior. Moreover, the signal structures of linear time-invariant (i.e. LTI) systems can be characterized by adjacency matrices of single-input single-output transfer functions describing the pairwise dependency between each ordered pair of manifest variables. So, for example, a system with N manifest variables, $w_i(t)$, $i = 1, 2, ..., N$ where $t \in \mathbb{R}$ or $t \in \mathbb{Z}$, depending on whether the signals are defined over continuous or discrete time, may be characterized by the behavior $(I - \bar{Q})w = 0$:

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix} = \begin{bmatrix} 0 & \bar{Q}_{12} & \dots & \bar{Q}_{1N} \\ \bar{Q}_{21} & 0 & \dots & \bar{Q}_{2N} \\ \vdots & & \ddots & \vdots \\ \bar{Q}_{N1} & \dots & \bar{Q}_{N(N-1)} & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix},$$

where the signal vector w may be characterized in either the time or frequency domain, and the hollow operator $\bar{Q}$ may correspondingly represent an impulse response matrix or a matrix of transfer functions, respectively. Note that all LTI systems have such a representation–even though $\bar{Q}$ having zeros on the diagonal and the $(I+\bar{Q})$ structure would appear to be somewhat special.

Sometimes a (possibly empty) subset of the manifest variables may not depend on any of the other variables, meaning the entire set of rows of corresponding entries in $\bar{Q}$ is zero. When this happens, we can always re-label those variables as $u_1, ..., u_m$, and the remaining manifest variables as $y_1, ..., y_p$, where $p + m = N$. Permuting variables then leads to a very structured characterization of $\bar{Q}$:

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} Q & P \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix},$$

where the variables u have the natural interpretation as *inputs* to the system, and y may be considered *outputs* of the system. Note that the inputs u may be controlled or uncontrolled, deterministic or stochastic, etc.–their only constraint is that they are not affected by the other manifest signals of the system.

In the case where some manifest variables do not depend on the others, as described above, the pair (Q, P) completely characterizes $\bar{Q}$, and thus the behavior of the system. We can see this characterization by rearranging the equation

$$\begin{aligned} y &= Qy + Pu \\ \Rightarrow \quad (I - Q)y &= Pu \\ \Rightarrow \quad y &= \left[(I - Q)^{-1}P\right] u \\ \Rightarrow \quad y &= Gu \text{ where } G = (I - Q)^{-1}P, \end{aligned}$$

revealing that the pair (Q, P) is, in fact, a kind of left factorization of the input-output map, G, whether characterized as the impulse response matrix operating via convolution, or the transfer function operating via multiplication. Note that $(I - Q)$ may not always be invertible; *well-posedness* of the resulting system representation, (Q, P), is intimately tied to the invertibility of $(I - Q)$ [3].

This pair, (Q, P) is called the *dynamical structure function*, or *DSF* [4], of the system, and it is our primary

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This material is based on research sponsored by the Department of Homeland Security, (DHS) Science and Technology Directorate, via contract number HSHQDC-15-C-B0056. All material in this paper represents the position of the authors and not necessarily that of DHS.

vehicle for representing dynamic networks in the input-output setting, although it is straightforward to extend the theory back to $\bar{Q}$ in the more general behavioral setting [5]. Indeed, we find the network structure associated with the DSF–or the system's *signal structure*–from the adjacency structure of $\bar{Q}$, where non-zero entries in (Q, P) identify the directed edges between the corresponding variables.

B. Abstractions and Realizations of Dynamic Networks

Although the ideas will be made precise in Section III, here we want to introduce the concepts of abstraction and realization of a dynamic network characterized by (Q, P). Loosely speaking, an abstraction of a dynamic network is another dynamic network with equivalent dynamic behavior, but less structural detail. A realization of a dynamic network, on the other hand, is another dynamic network with equivalent dynamic behavior but more structural detail.

In the literature, these ideas have been explored in various ways. For example, in [5]–[9], the DSF itself is presented as an abstraction of a particular state space model while simultaneously being a realization of the corresponding transfer function. In fact, we obtain different levels of abstraction specifically by modeling or hiding different numbers of system variables, leading to the notion of the DSF being a "partial structure representation" of the system. Likewise, in [10], [11], the concept of abstraction is called *immersion* and follows a similar notion of hiding a variable or set of variables to obtain a dynamically equivalent, but structurally less informative, model of the dynamic network. The work in [12] obtains abstractions by specifying $\hat{y} = Cy$, where y contains the outputs of the DSF above and C is a projection operator that specifies how those outputs are observed.

In general, however, the concept is easy to imagine if one starts with a particular state space model with full state measurements:

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bu(t) \\ y(t) & = & x(y). \end{array}$$

The corresponding *full-state* DSF (e.g. in the frequency domain) is then given by:

$$\begin{array}{rcl} Q(s) & = & \operatorname{diag}(sI - \operatorname{diag}(A))^{-1} \left[A - \operatorname{diag}(A)\right], \\ P(s) & = & \operatorname{diag}(sI - \operatorname{diag}(A))^{-1} B \end{array}$$

(where $\operatorname{diag}(A)$ is a square matrix the size of $A \in \mathbb{R}^{n \times n}$ containing only A's diagonal entries), since then $Y(s) = Q(s)Y(s) + P(s)U(s)$, where $Y(s)$ and $U(s)$ are the Laplace transforms of $y(t)$ and $u(t)$, respectively. Note that A and Q (and B and P) have the same sparsity pattern (with the exception that Q is hollow)–thus ensuring that the signal structure of this DSF is equivalent to the structural information defined by this particular state space model. This DSF is thus the most *refined* dynamical network model for this system.

At the other extreme, we could consider a different dynamic network model for the same system, but where $\tilde{Q} = 0$ and $\tilde{P} = G = (sI - A)^{-1}B$. It can be verified that both (Q, P) and $(\tilde{Q}, \tilde{P})$ describe the same dynamic behavior, since $(I - Q)^{-1}P = (I - \tilde{Q})^{-1}\tilde{P}$. Nevertheless, the $(0, G)$ representation is a very *coarse* representation, since it contains *no* structural information about the system beyond that in the transfer function itself. That is to say, $(0, G)$ is the dynamic network one obtains by abstracting away *every* variable y_i, $i = 1, ..., n$ in (Q, P). Thus, $(0, G)$ is the uniquely defined *most extreme* abstraction possible of (Q, P), while (Q, P) is one realization of many of $(0, G)$. More about abstractions and realizations will be detailed in Section III.

C. Semantics of Network Structures

The meaning of the signal structure encoded by a dynamical structure function has been studied thoroughly in [13], [14]. The important points to review for this work include the following:

- Signal structure can (and generally does) differ quite a bit from *subsystem structure*, the interconnection pattern of subsystems often described, for example, by block diagrams. It is not correct to interpret the elements or modules of a dynamic network corresponding to a specific edge or link in the signal structure as a subsystem since subsystems necessarily have distinct internal states while the modules of a DSF can (but don't need to) share internal state [5], [15].
- As a result of possible shared state in DSFs, the minimal order of a state space system necessary to realize a given DSF (or the *structural degree of the DSF*) is bounded below by the McMillan Degree of the corresponding transfer function G and bounded above by the sum of the McMillian Degrees of each module (or element) in Q and P [7], [16], [17].

D. Proper Networks and the Need for Well-Posedness

Although characterizing the signal structures of various classes of nonlinear systems is still an active research topic, the dynamical structure function is well-defined for any LTI system characterized by an arbitrary state-space description, (A, B, C, D) [13]. In the frequency domain, the resulting dynamical structure function will result in $P(s)$ being strictly proper if and only if $D = 0$, and $Q(s)$ is always strictly proper. One may wonder, then, why well-posedness is an issue for dynamical networks if the interconnection matrix $Q(s)$ is necessarily strictly proper.

The reason is that interconnections of systems, each system with its own well-defined signal structure, can result in a signal structure, and corresponding dynamical structure function, where $Q(s)$ is not strictly proper (see equation (13) in [18]). This happens with the feedback interconnection of two systems, each of which has a strictly proper $Q(s)$ but a non-strictly proper $P(s)$. The dynamical structure function of the closed-loop system can then be easily seen to have a non-strictly-proper $Q(s)$. This fact has motivated recent work in characterizing the nature of well-posedness for dynamical networks, c.f [3], [19].

E. Related Work and Contributions

Well-posedness of interconnected systems has been discussed since [20], and [21] gives conditions for the well-posedness for the interconnection of multiple (more than two) subsystems. Since the DSF is similar to an interconnection of subsystems, but semantically different [13], separate conditions for well-posedness of DSFs must be studied. The condition for well-posedness in this paper (Theorem 1) was initially stated in [22] and proven in [3]. Furthermore, in [3], it was shown that the notion of well-posedness becomes more nuanced as we consider abstractions of DSFs, that the necessary and sufficient conditions for the well-posedness of a DSF are not, in general, necessary or sufficient for the well-posedness of an abstraction. Stronger conditions for well-posedness were stated in [19], [23], [24], and in this work, we show that those stronger conditions, while not necessary for the well-posedness of a DSF, do provide sufficient conditions for the well-posedness of the DSF along with all of its abstractions (Theorem 5).

In this work, we build upon [3], focusing on an understanding of conditions of well-posedness on abstractions of DSFs. We provide a sufficient condition under which a particular abstraction of some DSF is well-posed (Theorem 3). We also provide sufficient conditions for the DSF to be well-posed assuming that its abstraction is well-posed (Theorem 4). Finally, we provide sufficient conditions for *all* of the abstractions of some DSF to be well-posed (Theorem 5).

II. Preliminaries

Much of this work centers around functions of the form $g(s) = \frac{n(s)}{d(s)}$, where $n(s)$ and $d(s)$ are polynomials in $s \in \mathbb{C}$. If the degree of $n(s)$ is no greater than the degree of $d(s)$, we say that $g(s)$ is *proper*. If the degree of $n(s)$ is strictly less than the degree of $d(s)$, then we say that $g(s)$ is *strictly proper*.

We are also concerned with matrices consisting of these functions of rational polynomials. We say that the matrix $G(s) = [q_{ij}(s)]$ is *proper* if every rational polynomial $q_{ij}(s)$ is proper, and likewise $G(s)$ is *strictly proper* if every $q_{ij}(s)$ is strictly proper.

We say that a square matrix $G(s)$ of rational polynomials is *invertible almost everywhere* if it is singular only for a finite choice of values $s \in \mathbb{C}$. For brevity, from this point forward, when we say that $G(s)$ is invertible, we mean that it is invertible almost everywhere.

The following result from [25] shows when a proper $G(s)$ has a proper inverse:

Proposition 1 *Let $G(s)$ be a square matrix of proper rational polynomials. Then the inverse of $G(s)$ exists and is proper (a) if and (b) only if $G(\infty) \triangleq \lim_{s\to\infty} G(s)$ is nonsingular.* □

The Schur Complement will also come into play later in this work. Let M be an arbitrary matrix partitioned such that

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{1}$$

Then we call $M/D = A + BD^{-1}C$ the *Schur Complement* of D in M. Given this definition, we state without proof (the proofs are either known or follow immediately from Proposition 1) a few results pertaining to Schur Complements.

Proposition 2 *Let M be a matrix partitioned according to (1). Then if any two of the following matrices are invertible, the third is as well: (i) M, (ii) D, or (iii) M/D.* □

Corollary 1 *Let $G(s)$ be a square matrix of rational polynomials partitioned according to (1). Then if any of the following has a proper inverse, the third has one as well: (i) $G(s)$, (ii) $D(s)$, or (iii) $G(s)/D(s)$.* □

We also include the following well-known identity for block matrix inversion:

Proposition 3 *Let M be a matrix partitioned according to (1), and suppose that D and $E = M/D$ are both invertible. Then*

$$M^{-1} = \begin{bmatrix} E^{-1} & -E^{-1}BD^{-1} \\ -D^{-1}CE^{-1} & D^{-1} + D^{-1}CE^{-1}BD^{-1} \end{bmatrix}. \tag{2}$$

□

Finally, we include one more result using Schur Complements and block matrix inversion which will be useful later in this work.

Corollary 2 *Let $M(s)$ be a matrix of rational polynomials partitioned according to (1). If $D(s)$ has an inverse (which is not necessarily proper) and $M(s)$ has a proper inverse, then $M(s)/D(s)$ has a proper inverse.* □

Proof By Proposition 2, we have that $M(s)/D(s)$ must be invertible. Furthermore, since $M(s)/D(s)$ and $D(s)$ both have inverses, by Proposition 3, we can express

$$M^{-1}(s) = \begin{bmatrix} \tilde{A}(s) & \tilde{B}(s) \\ \tilde{C}(s) & \tilde{D}(s) \end{bmatrix}, \tag{3}$$

with $\tilde{A}(s) = (M(s)/D(s))^{-1}$. Since $M^{-1}(s)$, by assumption, is proper, we have that $(M(s)/D(s))^{-1}$ must also be proper. ■

III. Abstractions of Networks

Suppose that we wish to cease to observe or model certain output signals in a network, thus reducing our set of manifest variables. Since we define links in the DSF as the interaction between manifest variables, independent from other manifest variables, the DSF must change in order to reflect this reduction in the number of output signals while preserving this interaction structure. We call this transformation an *abstraction*.

Definition 1 (Abstraction) Let Y be a set of output signals in some proper DSF $F = (Q, P)$. Let $Y_S \subset \{Y_1, Y_2, \ldots, Y_p\}$ be the set of output signals we choose to observe and model corresponding to the index set $S \subset \{1, 2, \ldots, p\}$ and with $Y_{\bar{S}}$ defining the remaining outputs that are no longer observed. Let $F_S = (Q_S, P_S)$ be the resulting network after we cease to observe the signals $Y_{\bar{S}}$. We call F_S an *abstraction* of F with respect to S. Let the mapping–called the *abstraction* *function*–from F to F_S, generated by abstracting away outputs $Y_{\bar{S}}$, be denoted by $F_S = (Q_S, P_S) = \mathcal{A}(F \mid S)$. □

For notational simplicity, we sometimes leave out P in the abstraction and write $Q_S = \mathcal{A}(Q \mid S)$.

To define the abstraction function $\mathcal{A}(F \mid S)$, let $F = (Q, P)$, then permute and rewrite the DSF such that

$$\begin{bmatrix} Y_S \\ Y_{\bar{S}} \end{bmatrix} = \begin{bmatrix} Q_{SS} & Q_{S\bar{S}} \\ Q_{\bar{S}S} & Q_{\bar{S}\bar{S}} \end{bmatrix} \begin{bmatrix} Y_S \\ Y_{\bar{S}} \end{bmatrix} + \begin{bmatrix} P_{SS} \\ P_{\bar{S}\bar{S}} \end{bmatrix} U. \tag{4}$$

Solving for $Y_{\bar{S}}$, we get

$$Y_{\bar{S}} = (I - Q_{\bar{S}\bar{S}})^{-1} Q_{\bar{S}S} + (I - Q_{\bar{S}\bar{S}})^{-1} P_{\bar{S}} U. \tag{5}$$

Plugging this back into the first row of (4) gives

$$Y_S = W Y_S + V U, \tag{6a}$$
$$W = Q_{SS} + Q_{S\bar{S}}(I - Q_{\bar{S}\bar{S}})^{-1} Q_{\bar{S}S}, \tag{6b}$$
$$V = P_{SS} + Q_{S\bar{S}}(I - Q_{\bar{S}\bar{S}})^{-1} P_{\bar{S}\bar{S}}. \tag{6c}$$

At this point, (6a) is nearly in the form of the signal structure representation of the abstraction. However, by definition of the signal structure, Q must be hollow, but W in general will not be hollow. To correct this, let

$$D_W \triangleq \text{diag}(W) = \text{diag}(Q_{S\bar{S}}(I - Q_{\bar{S}\bar{S}})^{-1} Q_{\bar{S}S}), \tag{7}$$

where the latter equality holds since Q_{SS} is hollow. Subtracting $D_W Y_S$ from both sides of (6a) gives

$$(I - D_W) Y_S = (W - D_W) Y_S + V U \qquad \Longrightarrow \tag{8}$$
$$Y_S = (I - D_W)^{-1}(W - D_W) Y_S + (I - D_W)^{-1} V U \tag{9}$$
$$\triangleq Q_S Y_S + P_S U. \tag{10}$$

Since Q_S is hollow, we have the signal structure representation of the abstraction $F_S = \mathcal{A}(F \mid S)$ given by $F_S = (Q_S, P_S)$.

With $F_S = \mathcal{A}(F \mid S)$ as the abstraction of some DSF F, we call F a *realization* of F_S. Realizations of DSFs need not be unique; in particular there could potentially exist infinitely many unique $\hat{F} \neq F$ such that $F_S = \mathcal{A}(\hat{F} \mid S)$.

For completeness, we also define the truncation function $\mathcal{T}$, which is similar to the abstraction function. Where the abstraction function preserves the dynamics of the outputs that are no longer modeled, the truncation function discards those dynamics. We have that, for some proper DSF $F = (Q, P)$ partitioned according to (4) (and where Q_{SS} and P_{SS} are defined in that equation), the truncation function is defined as

$$\mathcal{T}(F \mid S) \triangleq (Q_S, P_S) = (Q_{SS}, P_{SS}). \tag{11}$$

IV. Well-Posedness

We now turn our attention to providing conditions for well-posedness of DSFs and their abstractions.

A. Well-Posedness of DSFs

It is well-known [20], [21] that the feedback interconnection of two or more proper transfer functions can lead to algebraic inconsistencies–where signals either do not exist or are not unique given some choices of inputs and internal states. DSFs with proper–and not necessarily strictly proper–entries in Q may potentially result in a feedback interconnection of proper systems; thus we wish to characterize when a DSF has no such algebraic inconsistencies; i.e., when a DSF is *well-posed*.

We adapt the definition of well-posedness from [20], noting that some authors only use a subset of the conditions listed below[1].

Definition 2 (Well-Posedness) The dynamical structure function is *well-posed* if the following conditions are satisfied:

(a) The internal signals of all feedback loops in (Q, P), which are also the outputs $y_1, \ldots, y_p$, are uniquely defined for every choice of the realization state variables and external inputs.
(b) The internal signals of all feedback loops depend causally on the realization state variables and external inputs.
(c) The internal signals of all feedback loops depend continuously on the realization state variables and external inputs.
(d) Small changes in the model should not result in any feedback loop that does not satisfy the previous conditions.

A DSF that is not well-posed is called *ill-posed*. □

Given this definition, we can state the following theorem about the conditions for well-posedness of a DSF, relegating the reader to [3] for the proof.

Theorem 1 *Let $F = (Q(s), P(s))$ be a proper DSF. Then the following are equivalent:*
(T1) F is well-posed.
(T2) $(I - Q(s))$ has a proper inverse.
(T3) $(I - Q(\infty)) \triangleq \lim_{s\to\infty}(I - Q(s))$ is invertible.

It turns out that well-posedness is only an issue with DSFs that are proper but not strictly proper. We formally state this here.

Corollary 3 *Let $F = (Q(s), P(s))$ be a proper DSF such that $Q(s)$ is strictly proper. Then F is well-posed.* □

Proof Let $p(s) = \frac{n(s)}{d(s)}$ be a strictly proper rational polynomial. Then $\lim_{s\to\infty} \frac{n(s)}{d(s)} = 0$. Since $Q(s)$ is strictly proper by assumption, we have that

$$(I - Q(\infty)) = \lim_{s\to\infty}(I - Q(s)) = I,$$

which is always invertible. Thus by Theorem 1, F is well-posed. ■

[1]The works [20] and [26] use all four conditions, where [21] and [27] only use conditions (a-c), [28] uses only conditions (a-b), and [29] only uses condition (a).

B. Well-Posedness of DSF Abstractions

The notion of well-posedness becomes more nuanced as we consider abstractions. The purpose of this section is to provide sufficient conditions under which realizations and abstractions are both proper and well-posed. We begin with a couple of definitions.

Definition 3 (Strong Well-Posedness) Let $F = (Q, P)$ be a proper DSF with outputs $Y_1, \ldots, Y_p$ and let $S \subset \{1, \ldots, p\}$. Then we say that F is *strongly well-posed* with respect to S if $F_S = \mathcal{A}(F \mid S)$ is both proper *and* well-posed. $\square$

Definition 4 (Total Well-Posedness) Let $F = (Q, P)$ be a proper DSF with outputs $Y_1, \ldots, Y_p$ and let $\mathscr{S} = \mathscr{P}(\{1, \ldots, p\})$ be the power set of indexing all possible subsets of the outputs. Then we say that F is *totally well-posed* if F is strongly well-posed with respect to every $S \in \mathscr{S}$. In other words, F is totally well-posed if every abstraction is proper and well-posed. $\square$

We can now state the necessary and sufficient conditions for some DSF F to be strongly well-posed:

Theorem 2 *Let $F = (Q, P)$ be a proper DSF with outputs $Y_1, \ldots, Y_p$ and let $S \subset \{1, \ldots, p\}$. Also let W, V, and D_W be defined as given in Section III. Then F is strongly well-posed with respect to S if and only if the following exist and are proper:*

- $(I - D_W)^{-1}(W - D_W)$
- $(I - D_W)^{-1}V$
- $(I - W)^{-1}(I - D_W)$ $\square$

PROOF Let $F_S = \mathcal{A}(F \mid S)$. The first two conditions above are necessary and sufficient for F_S to exist and be proper. We have that

$$(I - Q_S) = I - (I - D_W)^{-1}(W - D_W),$$

thus

$$\begin{aligned}(I - D_W)(I - Q_S) &= (I - D_W) - (W - D_W)\\ &= (I - W),\end{aligned}$$

giving

$$(I - Q_S)^{-1} = (I - W)^{-1}(I - D_W). \tag{12}$$

Hence the third is necessary and sufficient for the F_S to be well-posed. $\blacksquare$

We are now prepared to prove the following intermediate result which will be helpful later:

Lemma 1 *Let $F = (Q, P)$ be a proper DSF with outputs $Y_1, \ldots, Y_p$ and let $S \subset \{1, \ldots, p\}$. Also let $Q_{\bar{S}\bar{S}}$ and D_W be defined as given in Section III. If $(I - Q_{\bar{S}\bar{S}})$ is invertible (though not necessarily with a proper inverse) and if $\mathcal{T}(F \mid \bar{S} \cup i)$ for each $i \in S$ is well-posed, then $(I - D_W)$ has a proper inverse.* $\square$

PROOF Let U_i be the ith row of $Q_{S\bar{S}}$ and let V_i be the ith column of $Q_{\bar{S}S}$. Then $\mathcal{T}(F \mid \bar{S} \cup i) \triangleq (Q_{Si}, P_{Si})$ has

$$(I - Q_{Si}) = \begin{bmatrix} 1 & -U_i \\ -V_i & (I - Q_{\bar{S}\bar{S}}) \end{bmatrix}. \tag{13}$$

Thus, if $\mathcal{T}(F \mid \bar{S} \cup i)$ is well-posed for some $i \in S$, we have that that $(I - Q_{Si})$ has a proper inverse. And since $(I - Q_{\bar{S}\bar{S}})$ is invertible, by Corollary 2, we have that $(I - Q_{Si})/(I - Q_{\bar{S}\bar{S}}) = 1 - U_i(I - Q_{\bar{S}\bar{S}})^{-1}V_i$ has a proper inverse. However, we also have that

$$\begin{aligned}I - D_W &= I - \mathrm{diag}(Q_{S\bar{S}}(I - Q_{\bar{S}\bar{S}})^{-1}Q_{\bar{S}S})\\ &= 1 - U_1(I - Q_{\bar{S}\bar{S}})^{-1}V_1 \oplus \cdots\\ &\qquad \oplus 1 - U_{\tilde{p}}(I - Q_{\bar{S}\bar{S}})^{-1}V_{\tilde{p}},\end{aligned} \tag{14}$$

where $\tilde{p} = |\bar{S}|$. Thus we have that $(I - D_W)$ must also have a proper inverse, as desired. $\blacksquare$

We can now provide a set of sufficient conditions on well-posedness assuming that either the abstraction or the realization is well-posed.

Theorem 3 *Let $F = (Q, P)$ be a proper DSF with an abstraction $F_S = (Q_S, P_S) = \mathcal{A}(F \mid S)$. If F, $\mathcal{T}(F \mid \bar{S})$, and $\mathcal{T}(F \mid \bar{S} \cup i)$ for each $i \in S$ are well-posed, then F_S is both proper and well-posed (i.e.; F is strongly well-posed according to Definition 3).* $\square$

PROOF Suppose that F, $\mathcal{T}(F \mid \bar{S})$, and $\mathcal{T}(F \mid \bar{S} \cup i)$ for each $i \in S$ are all well-posed. Then $(I - Q)$ and $(I - Q_{\bar{S}\bar{S}})$ have proper inverses. Thus $W = Q_{SS} + Q_{SS} + Q_{S\bar{S}}(I - Q_{\bar{S}\bar{S}})^{-1}Q_{\bar{S}S}$, D_W, and $(W - D_W)$ are all proper. By Corollary 1, we have that $(I - W) = (I - Q)/(I - Q_{\bar{S}\bar{S}})$ also has a proper inverse. And by Lemma 1, we have that $(I - D_W)$ also has a proper inverse.

Since the products of proper matrices of rational polynomials is also proper, we get that the products $V = P_{SS} + Q_{S\bar{S}}(I - Q_{\bar{S}\bar{S}})^{-1}P_{\bar{S}\bar{S}}$, $(I - D_W)^{-1}(W - D_W)$, $(I - D_W)^{-1}V$, and $(I - D_W)^{-1}(I - W)$ all exist and are proper. Thus, by Theorem 2, we have that F_S must also be well-posed. $\blacksquare$

Theorem 4 *Let $F = (Q, P)$ be a proper DSF with an abstraction $F_S = (Q_S, P_S) = \mathcal{A}(F \mid S)$. If F_S, $\mathcal{T}(F \mid \bar{S})$, and $\mathcal{T}(F \mid \bar{S} \cup i)$ for each $i \in S$ are well-posed, then F is both proper and well-posed.* $\square$

PROOF Suppose that F_S, $\mathcal{T}(F \mid \bar{S})$, and $\mathcal{T}(F \mid \bar{S} \cup i)$ for each $i \in S$ are all well-posed. Then, by Theorem 2, we have that $Q_S = (I - D_W)^{-1}(W - D_W)$, $P_S = (I - D_W)^{-1}V$, and $(I - Q_S)^{-1} = (I - W)^{-1}(I - D_W)$ all exist and are proper. Furthermore, by Lemma 1, we also have that $(I - D_W)^{-1}$ exists and is proper. Since $(I - W)^{-1} = (I - Q_S)^{-1}(I - D_W)^{-1}$ is the product of proper matrices, $(I - W)$ is also proper. And since both $(I - Q_{\bar{S}\bar{S}})$ and $(I - W) = (I - Q)/(I - Q_{\bar{S}\bar{S}})$ have proper inverses, by Corollary 1, $(I - Q)$ must also have a proper inverse; therefore by Theorem 1, $F = (Q, P)$ is well-posed. $\blacksquare$

Remark 1 Recall that Definition 3 (strong well-posedness) requires that the abstracted network be both proper and well-posed. In both Theorems 3 and 4, the fact that both $(I-Q_{\bar{S}\bar{S}})$ and $(I-D_W)$ exist, are proper, and have proper inverses ensures that the abstraction or the realization is proper given that the other is also proper. Thus, Theorems 3 and 4 can be restated as follows: if you can guarantee that both the DSF and its abstraction are proper, then well-posedness of one implies well-posedness of the other. □

We are now prepared to give sufficient conditions under which F is totally well-posed.

Theorem 5 *Let $F=(Q,P)$ be a proper DSF. Then F is totally well-posed if every principal submatrix of $(I-Q)$ has a proper inverse.* □

PROOF Suppose that every principal submatrix of $(I-Q)$ has a proper inverse. Since, for any choice of S, $(I-Q)$, $(I-Q_{\bar{S}\bar{S}})$, and $(I-Q_{Si})$ for $i \in S$ are all principal sub-matrices of $(I-Q)$, we have that they all have proper inverses; hence F, $\mathcal{T}(F \mid \bar{S})$ and $\mathcal{T}(F \mid S \cup i)$ for $i \in S$ are all well-posed. Thus by Theorem 3, every abstraction $F_S = \mathcal{A}(F \mid S)$ of F must also be well-posed, and F by definition is totally well-posed. ■

Corollary 4 *Let $F=(Q,P)$ be a proper DSF where Q is strictly proper. Then F and every abstraction of F is well-posed.* □

PROOF Follows immediately from Theorem 1 and Theorem 5 and the fact that, when Q is strictly proper, $\lim_{s\to\infty}(I-Q)=I$ is non-singular, as are all of its principle submatrices. ■

Total well-posedness thus becomes a stronger condition of well-posedness, guaranteeing that the network and every possible network abstraction is well-posed; thus total well-posedness may be a more reasonable assumption than well-posedness in many applications. For instance, [19], [23] state the conditions of Theorem 5 as the conditions for well-posedness.

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