

Characterizing the Structure of Oscillating Systems

Gustavo Rodriguez

Advisor: Sean Warnick



Information & Decision Algorithms Laboratories

Outline

- Background: Studying the Structure of a Linear Time Invariant System
- Motivation: Many systems oscillate, so current tools won't work
- First Results: Lifting a Time-Varying System
- Next Idea: Decoupled sampling
- Conclusions
- Future Work

Structure of a Linear Time-Invariant System

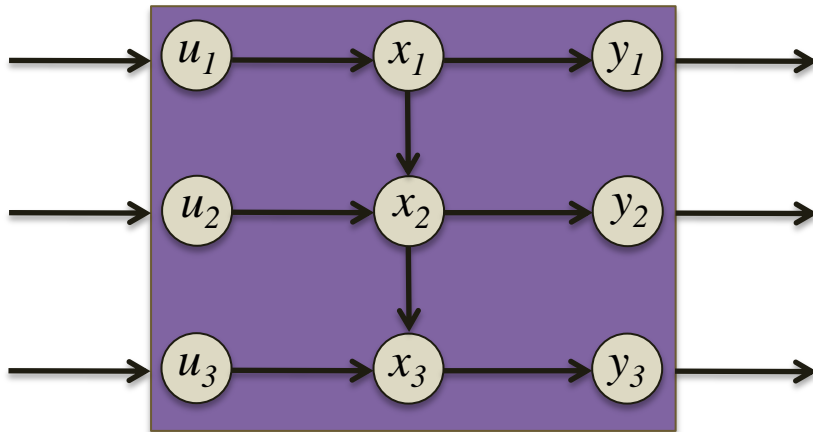
- Outline
- ▶ Background
- Motivation
- First Result
- Next Idea
- Conclusions
- Future Work

Two Representations:

- State Space Realization
 - Physical interconnections
- Transfer Function
 - Input-output relations

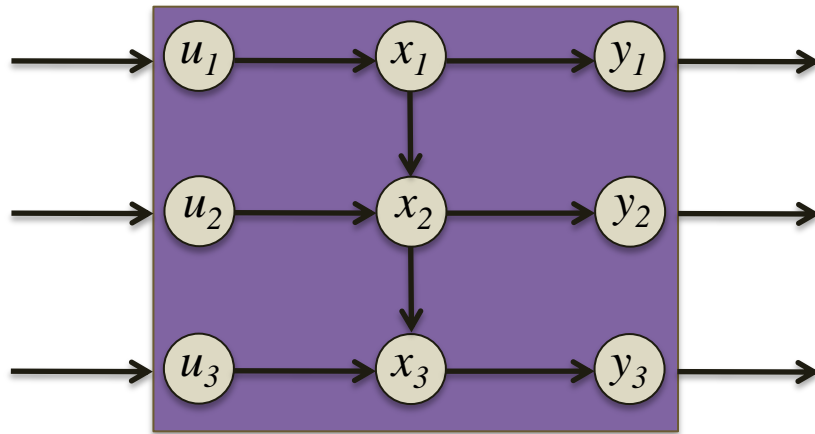
Structure of a Linear Time-Invariant System

- Outline
- ▶ Background
- Motivation
- First Result
- Next Idea
- Conclusions
- Future Work



Structure of a Linear Time-Invariant System

- ▶ Outline
- ▶ Background
- Motivation
- First Result
- Next Idea
- Conclusions
- Future Work



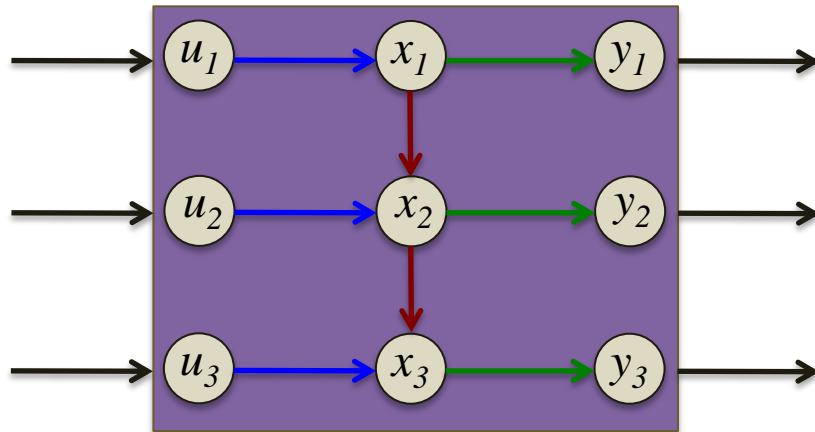
State Space Realization

\dot{x}_1	$=$	$-1x_1$	$+0x_2$	$+0x_3$	$+u_1$
\dot{x}_2	$=$	$-2x_1$	$-3x_2$	$+0x_3$	$+u_2$
\dot{x}_3	$=$	$0x_1$	$-2x_2$	$-3x_3$	$+u_3$
y_1	$=$	$1x_1$	$+0x_2$	$+0x_3$	
y_2	$=$	$0x_1$	$+1x_2$	$+0x_3$	
y_3	$=$	$0x_1$	$+0x_2$	$+1x_3$	



Structure of a Linear Time-Invariant System

- Outline
- ▶ Background
- Motivation
- First Result
- Next Idea
- Conclusions
- Future Work



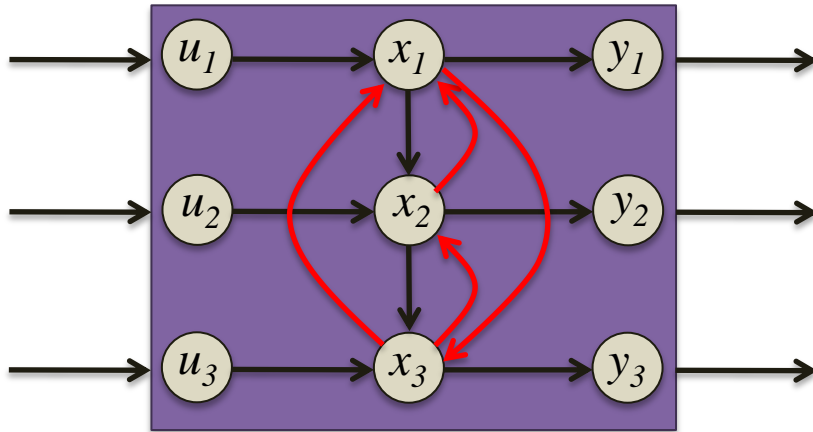
State Space Realization

$$\begin{aligned} \dot{\hat{x}}_1 &= \begin{bmatrix} -1 & 0 & 0 \\ -2 & -3 & 0 \\ 0 & -2 & -3 \end{bmatrix} \hat{x} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} u \\ \hat{y}_1 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \hat{x} \end{aligned}$$



Structure of a Linear Time-Invariant System

- Outline
- ▶ Background
- Motivation
- First Result
- Next Idea
- Conclusions
- Future Work

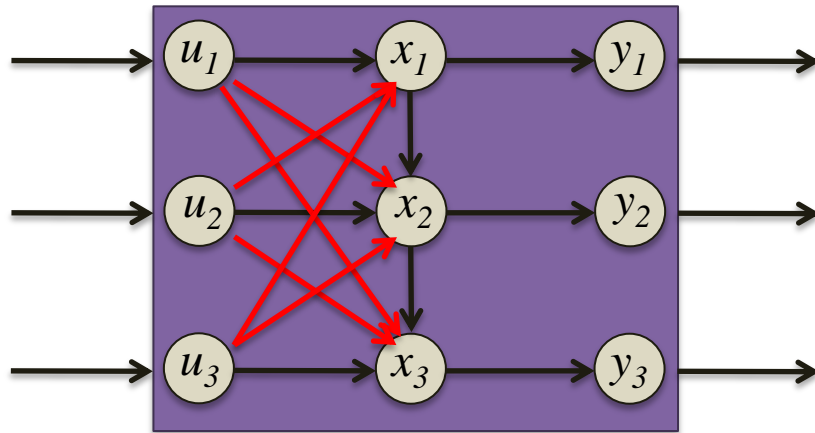


State Space Realization

\dot{x}_1	$=$	$-1x_1$	$+0x_2$	$+0x_3$	$+u_1$
\dot{x}_2	$=$	$-2x_1$	$-3x_2$	$+0x_3$	$+u_2$
\dot{x}_3	$=$	$+0x_1$	$-2x_2$	$-3x_3$	$+u_3$
y_1	$=$	$1x_1$	$+0x_2$	$+0x_3$	
y_2	$=$	$0x_1$	$+1x_2$	$+0x_3$	
y_3	$=$	$0x_1$	$+0x_2$	$+1x_3$	

Structure of a Linear Time-Invariant System

- ▶ Outline
- ▶ Background
- Motivation
- First Result
- Next Idea
- Conclusions
- Future Work



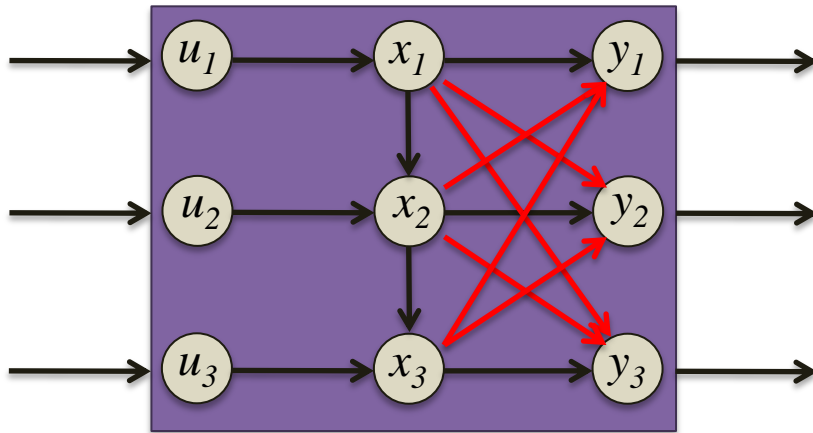
State Space Realization

\dot{x}_1	$=$	-1	0	0	x_1	$+$	1	0	0	u_1
\dot{x}_2	$=$	-2	-3	0	x_2	$+$	0	1	0	u_2
\dot{x}_3	$=$	0	-2	-3	x_3	$+$	0	0	1	u_3

y_1	$=$	1	0	0	x_1
y_2	$=$	0	1	0	x_2
y_3	$=$	0	0	1	x_3

Structure of a Linear Time-Invariant System

- ▶ Outline
- ▶ Background
- Motivation
- First Result
- Next Idea
- Conclusions
- Future Work



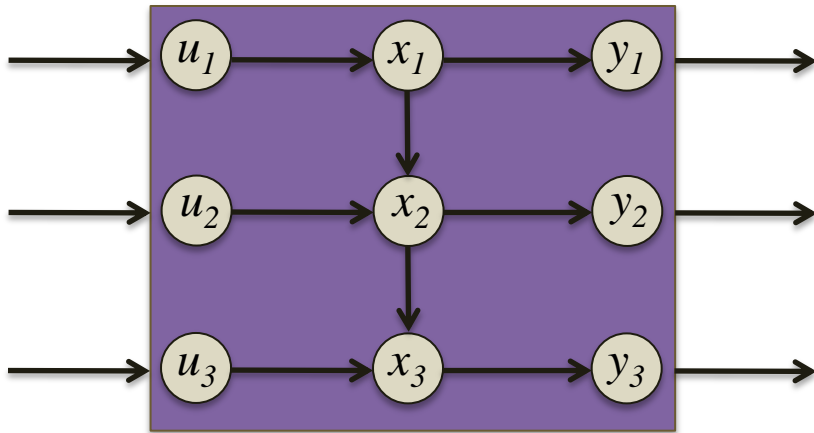
State Space Realization

$$\begin{aligned}
 \dot{x}_1 &= -x_1 + u_1 \\
 \dot{x}_2 &= -2x_2 + u_2 \\
 \dot{x}_3 &= -2x_3 + u_3
 \end{aligned}$$

$$\begin{aligned}
 y_1 &= x_1 + x_2 + x_3 \\
 y_2 &= x_1 + x_2 \\
 y_3 &= x_1 + x_2 + x_3
 \end{aligned}$$

Structure of a Linear Time-Invariant System

- Outline
- ▶ Background
 - Motivation
 - First Result
 - Next Idea
 - Conclusions
 - Future Work

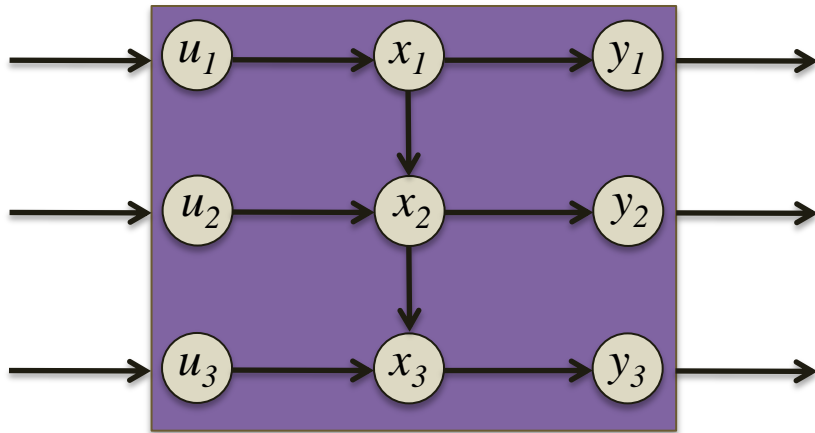


State Space Realization

$$\begin{aligned}
 \dot{x}_1 &= -x_1 + u_1 \\
 \dot{x}_2 &= -2x_2 + x_1 + u_2 \\
 \dot{x}_3 &= -2x_3 + x_2 + u_3 \\
 y_1 &= x_1 \\
 y_2 &= x_2 \\
 y_3 &= x_3
 \end{aligned}$$

Structure of a Linear Time-Invariant System

- Outline
- ▶ Background
- Motivation
- First Result
- Next Idea
- Conclusions
- Future Work



State Space Realization

$$\begin{matrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ y_1 \\ y_2 \\ y_3 \end{matrix} = \begin{bmatrix} -1 & 0 & 0 \\ -2 & -3 & 0 \\ 0 & -2 & -3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} u_1 \\ u_2 \\ u_3 \end{matrix}$$



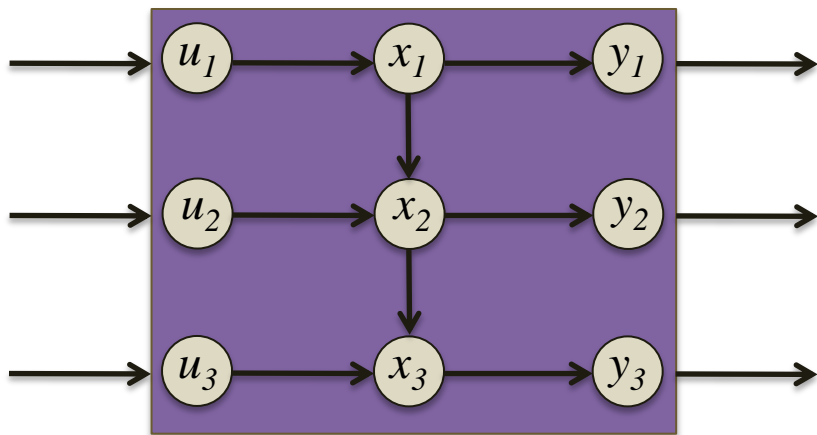
Transfer Function

$$\begin{matrix} Y_1 \\ Y_2 \\ Y_3 \end{matrix} = \begin{bmatrix} \frac{1}{s+1} & 0 & 0 \\ \frac{-2}{(s+1)(s+3)} & \frac{1}{s+3} & 0 \\ \frac{4}{(s+1)(s+3)^2} & \frac{-2}{(s+3)^2} & \frac{1}{s+3} \end{bmatrix} \begin{matrix} U_1 \\ U_2 \\ U_3 \end{matrix}$$

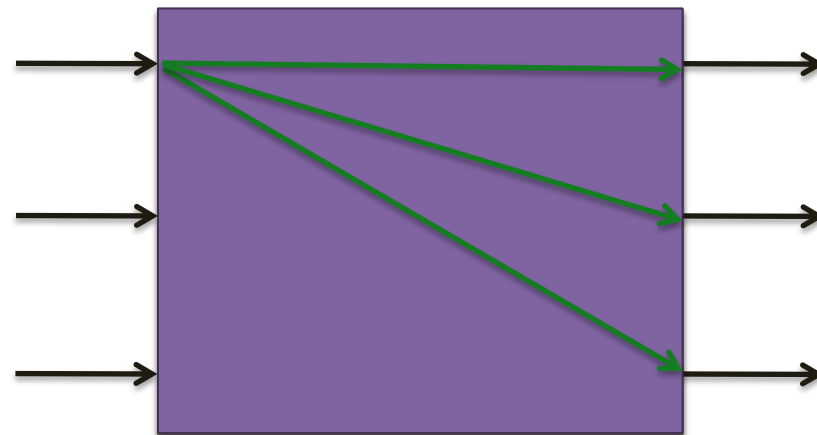


Structure of a Linear Time-Invariant System

- Outline
- ▶ Background
- Motivation
- First Result
- Next Idea
- Conclusions
- Future Work



State Space Realization



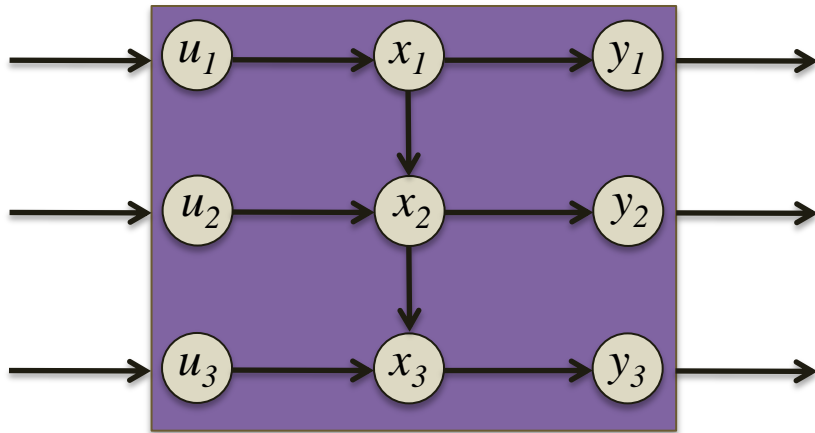
Transfer Function

$$\begin{array}{c}
 \dot{x}_1 \\
 \dot{x}_2 \\
 \dot{x}_3 \\
 y_1 \\
 y_2 \\
 y_3
 \end{array}
 =
 \begin{array}{ccc|ccc}
 -1 & 0 & 0 & 1 & 0 & 0 \\
 -2 & -3 & 0 & 0 & 1 & 0 \\
 0 & -2 & -3 & 0 & 0 & 1 \\
 \hline
 1 & 0 & 0 & 1 & 0 & 0 \\
 0 & 1 & 0 & 0 & 1 & 0 \\
 0 & 0 & 1 & 0 & 0 & 1
 \end{array}
 \begin{array}{c}
 x_1 \\
 x_2 \\
 x_3 \\
 u_1 \\
 u_2 \\
 u_3
 \end{array}$$

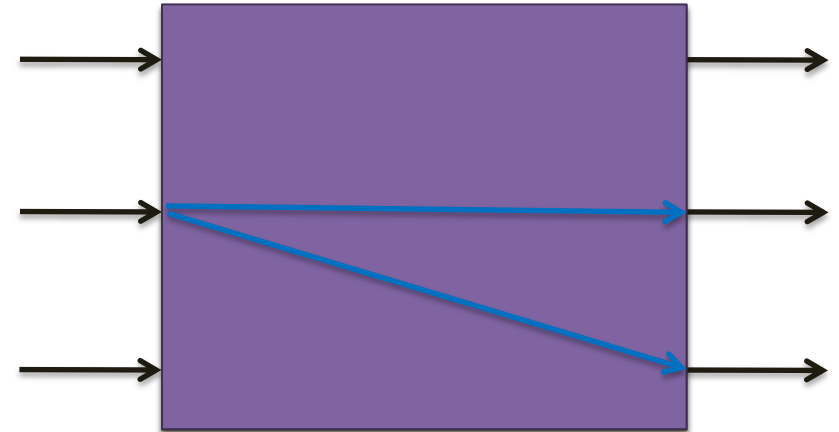
$$\begin{array}{c}
 Y_1 \\
 Y_2 \\
 Y_3
 \end{array}
 =
 \begin{array}{ccc|ccc}
 \frac{1}{s+1} & 0 & 0 & U_1 & U_2 & U_3 \\
 \frac{-2}{(s+1)(s+3)} & \frac{1}{s+3} & 0 & & & \\
 \frac{4}{(s+1)(s+3)^2} & \frac{-2}{(s+3)^2} & \frac{1}{s+3} & & &
 \end{array}$$

Structure of a Linear Time-Invariant System

- Outline
- ▶ Background
- Motivation
- First Result
- Next Idea
- Conclusions
- Future Work



State Space Realization



Transfer Function

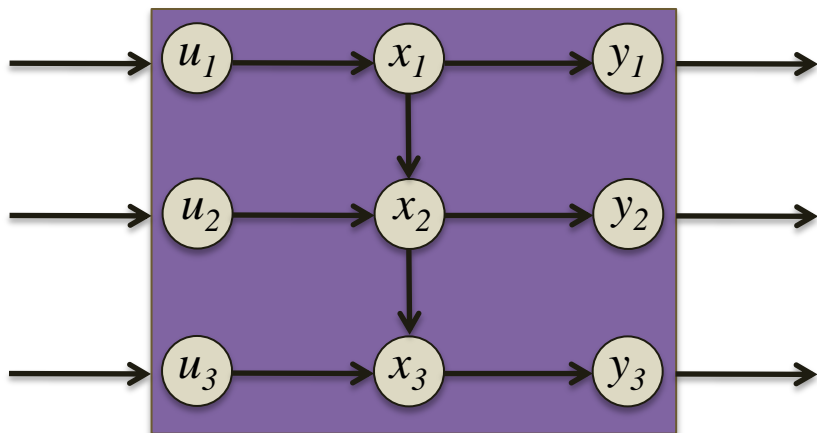
$$\begin{array}{c}
 \dot{x}_1 \\
 \dot{x}_2 \\
 \dot{x}_3 \\
 y_1 \\
 y_2 \\
 y_3
 \end{array}
 =
 \begin{array}{ccc|ccc}
 -1 & 0 & 0 & x_1 & u_1 \\
 -2 & -3 & 0 & x_2 & u_2 \\
 0 & -2 & -3 & x_3 & u_3 \\
 1 & 0 & 0 & x_1 & \\
 0 & 1 & 0 & x_2 & \\
 0 & 0 & 1 & x_3 &
 \end{array}$$

$$\begin{array}{c}
 Y_1 \\
 Y_2 \\
 Y_3
 \end{array}
 =
 \begin{array}{ccc|ccc}
 \frac{1}{s+1} & 0 & 0 & U_1 \\
 \frac{-2}{(s+1)(s+3)} & \frac{1}{s+3} & 0 & U_2 \\
 \frac{4}{(s+1)(s+3)^2} & \frac{-2}{(s+3)^2} & \frac{1}{s+3} & U_3
 \end{array}$$

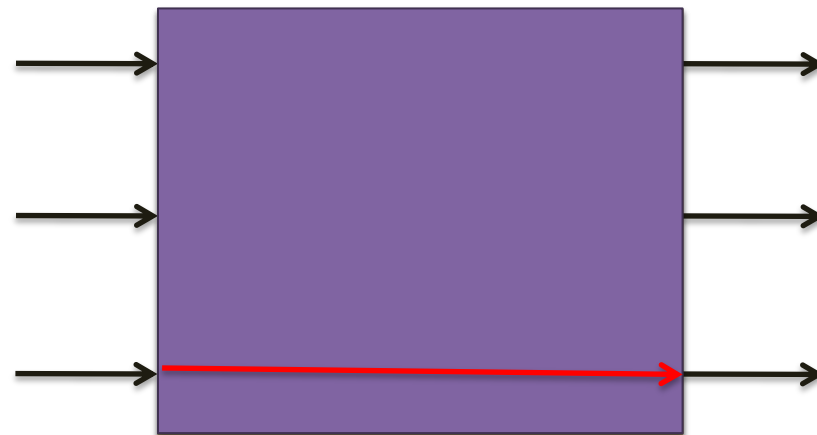


Structure of a Linear Time-Invariant System

- ▶ Outline
- ▶ Background
- Motivation
- First Result
- Next Idea
- Conclusions
- Future Work



State Space Realization



Transfer Function

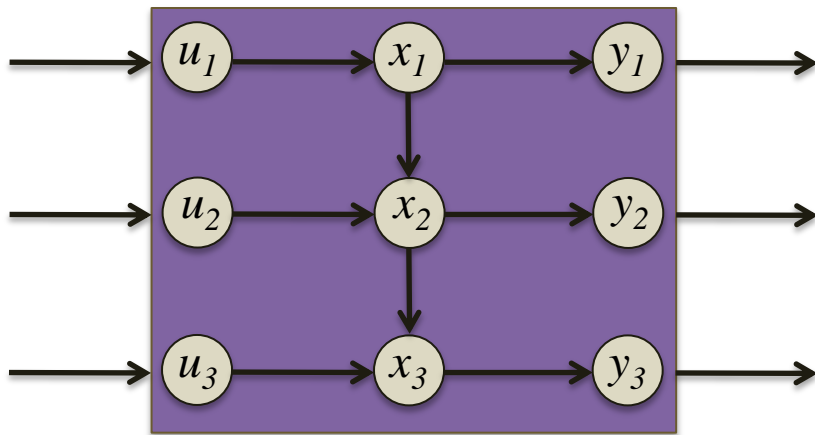
$$\begin{matrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ y_1 \\ y_2 \\ y_3 \end{matrix} = \begin{bmatrix} -1 & 0 & 0 \\ -2 & -3 & 0 \\ 0 & -2 & -3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} u_1 \\ u_2 \\ u_3 \end{matrix}$$

$$\begin{matrix} Y_1 \\ Y_2 \\ Y_3 \end{matrix} = \begin{bmatrix} \frac{1}{s+1} & 0 \\ \frac{-2}{(s+1)(s+3)} & \frac{1}{s+3} \\ \frac{4}{(s+1)(s+3)^2} & \frac{-2}{(s+3)^2} \end{bmatrix} \begin{matrix} U_1 \\ U_2 \\ U_3 \end{matrix}$$

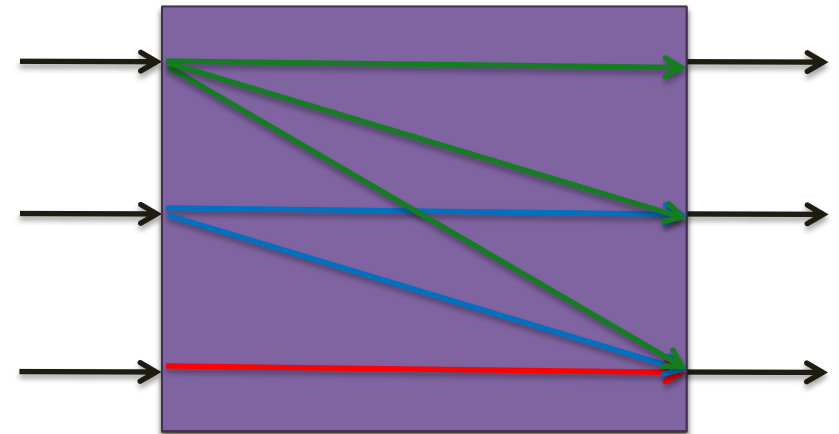


Structure of a Linear Time-Invariant System

- Outline
- ▶ Background
- Motivation
- First Result
- Next Idea
- Conclusions
- Future Work



State Space Realization



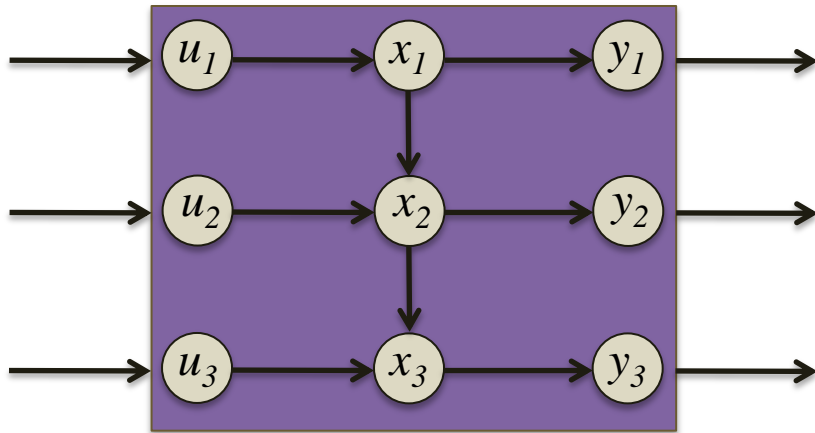
Transfer Function

$$\begin{aligned} \dot{x}_1 &= -x_1 + u_1 \\ \dot{x}_2 &= -2x_2 + x_1 + u_2 \\ \dot{x}_3 &= -2x_3 + x_2 + u_3 \\ y_1 &= x_1 \\ y_2 &= x_2 \\ y_3 &= x_3 \end{aligned}$$

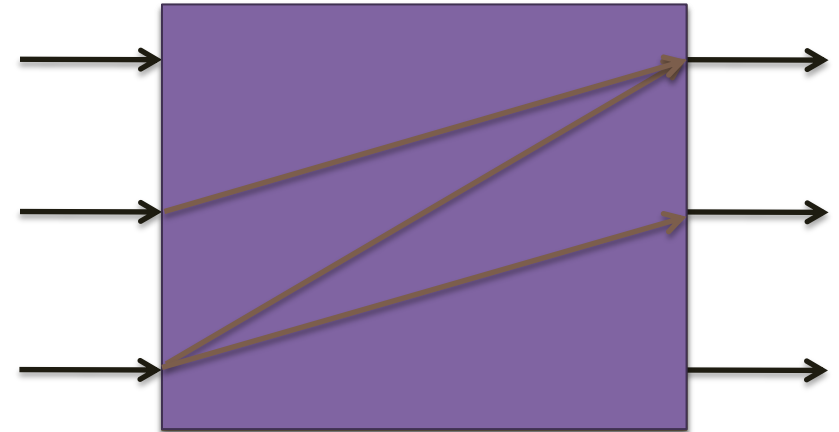
$$\begin{aligned} Y_1 &= \frac{1}{s+1} U_1 \\ Y_2 &= \frac{-2}{(s+1)(s+3)} U_1 + \frac{1}{s+3} U_2 \\ Y_3 &= \frac{4}{(s+1)(s+3)^2} U_1 - \frac{2}{(s+3)^2} U_2 + \frac{1}{s+3} U_3 \end{aligned}$$

Structure of a Linear Time-Invariant System

- ▶ Outline
- ▶ Background
- Motivation
- First Result
- Next Idea
- Conclusions
- Future Work



State Space Realization



Transfer Function

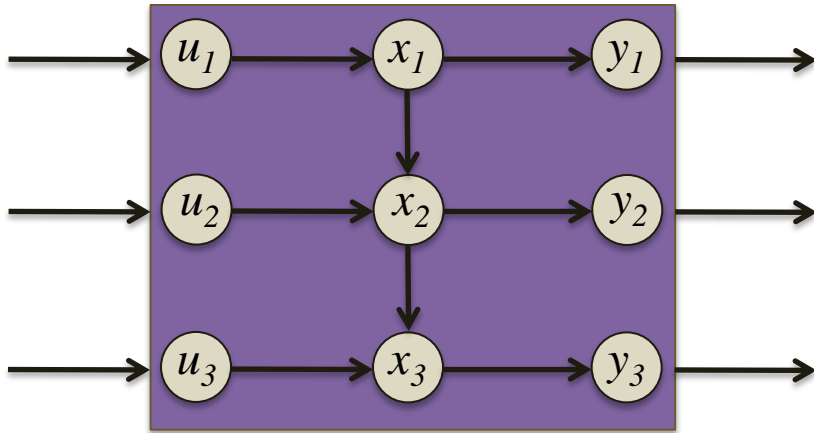
\dot{x}_1	$=$	-1	0	0	x_1	$+$	1	0	0	u_1
\dot{x}_2	$=$	-2	-3	0	x_2	$+$	0	1	0	u_2
\dot{x}_3	$=$	0	-2	-3	x_3	$+$	0	0	1	u_3
y_1	$=$	1	0	0	x_1					
y_2	$=$	0	1	0	x_2					
y_3	$=$	0	0	1	x_3					

Y_1	$=$	$\frac{1}{s+1}$	0	0	U_1
Y_2	$=$	$\frac{-2}{(s+1)(s+3)}$	$\frac{1}{s+3}$	0	U_2
Y_3	$=$	$\frac{4}{(s+1)(s+3)^2}$	$\frac{-2}{(s+3)^2}$	$\frac{1}{s+3}$	U_3



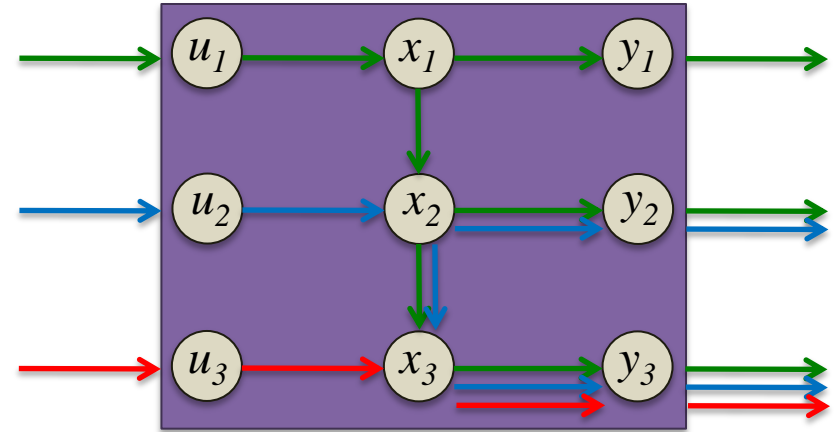
Structure of a Linear Time-Invariant System

- Outline
- ▶ Background
- Motivation
- First Result
- Next Idea
- Conclusions
- Future Work



State Space Realization

$$\begin{aligned} \dot{x}_1 &= -x_1 + u_1 \\ \dot{x}_2 &= -2x_2 - x_1 + u_2 \\ \dot{x}_3 &= -3x_3 - x_2 + u_3 \\ y_1 &= x_1 \\ y_2 &= x_2 \\ y_3 &= x_3 \end{aligned}$$



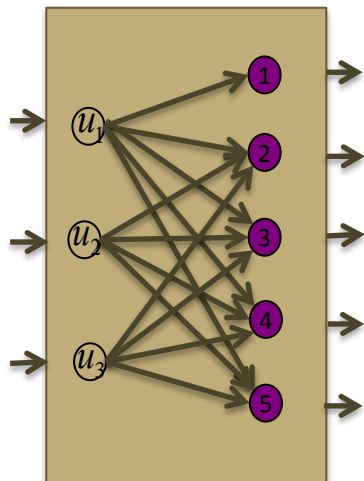
Transfer Function

$$\begin{aligned} Y_1 &= \frac{1}{s+1} U_1 \\ Y_2 &= \frac{-2}{(s+1)(s+3)} U_1 + \frac{1}{s+3} U_2 \\ Y_3 &= \frac{4}{(s+1)(s+3)^2} U_1 - \frac{2}{(s+3)^2} U_2 + \frac{1}{s+3} U_3 \end{aligned}$$

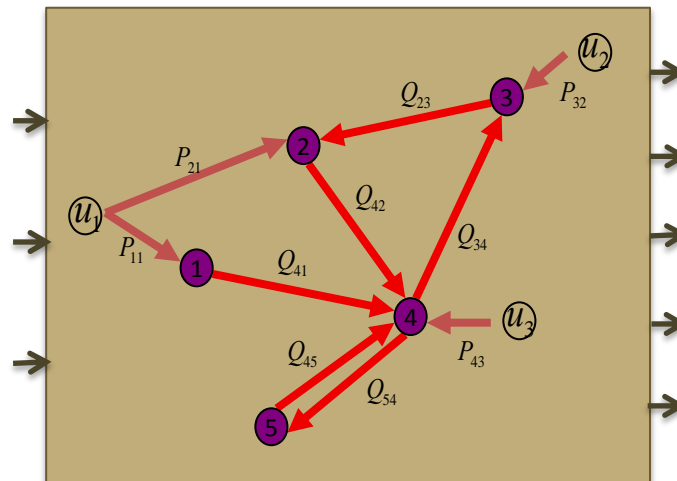
Dynamical Structure Functions

- Outline
- ▶ Background
- Motivation
- First Result
- Next Idea
- Conclusions
- Future Work

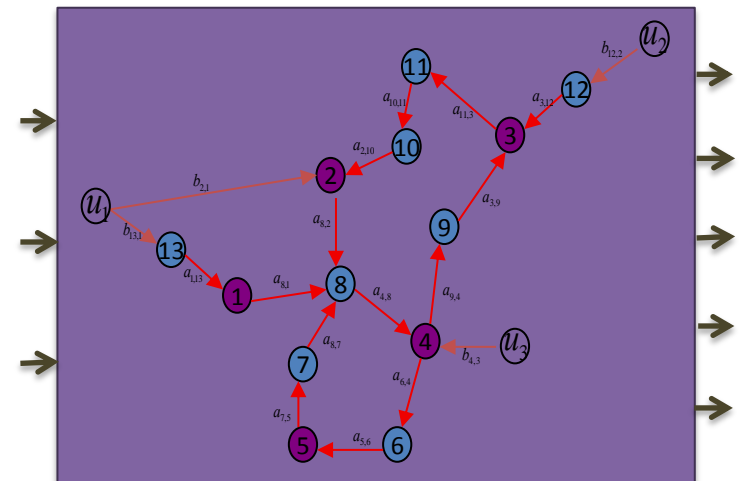
- Another way of representing a system
- Input-Output & Output-Output relations.
- More specific than the Transfer Function representation, but less specific than the State Space Realization.



Transfer Function



Dynamical Structure Function



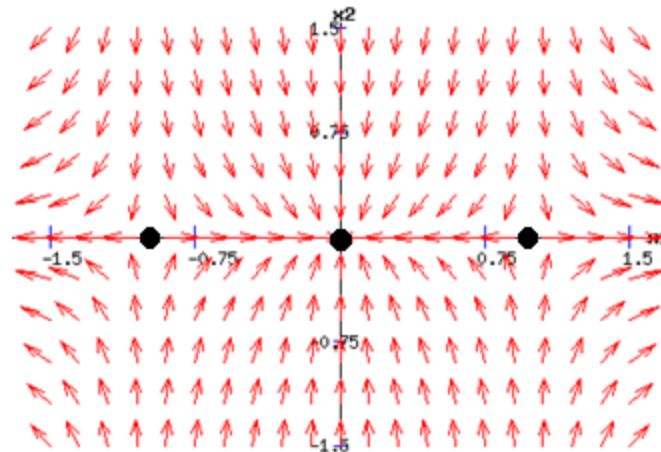
State Space Realization



Chemical Reaction Networks & Biological Systems

- Outline
- Background
- ▶ Motivation
- First Result
- Next Idea
- Conclusions
- Future Work

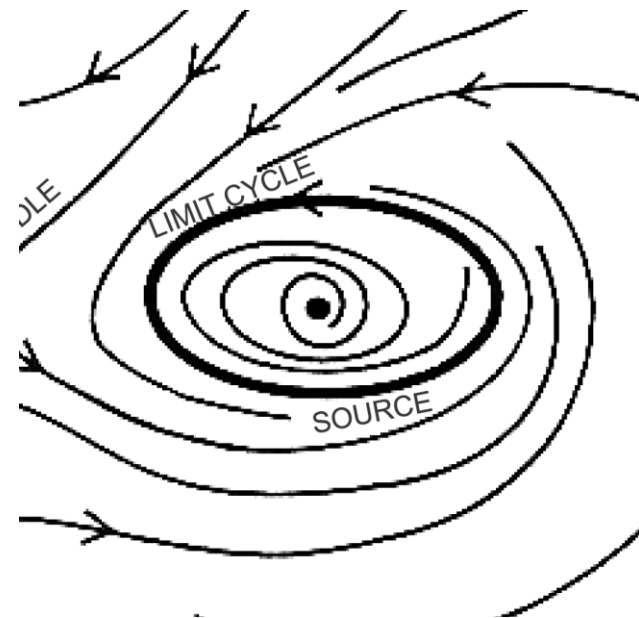
- These systems are actually nonlinear
 - Linearize near equilibrium point
 - Lyapunov: Structure of a linear system is the same structure of a nonlinear system near equilibrium point.
- Example: Homeostasis



Many Systems Oscillate

- Outline
- Background
- Motivation
- First Result
- Next Idea
- Conclusions
- Future Work

- Many biological systems oscillate
- Examples:
 - Breathing
 - Cardiovascular system
- These systems have no equilibrium point but rather a **limit cycle**
 - Linearize around limit cycle
 - Periodically Time Varying
- How to study structure?



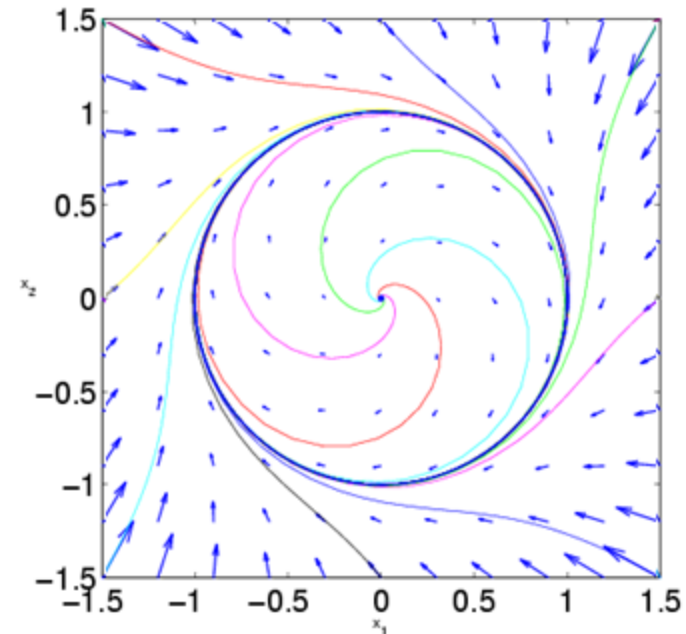
Example: Electronic Oscillator

- Outline
- Background
- Motivation
- First Result
- Next Idea
- Conclusions
- Future Work

- We want to study the structure of this system

$$\frac{dx_1}{dt} = x_2 + x_1(1 - x_1^2 - x_2^2)$$

$$\frac{dx_2}{dt} = -x_1 + x_2(1 - x_1^2 - x_2^2)$$



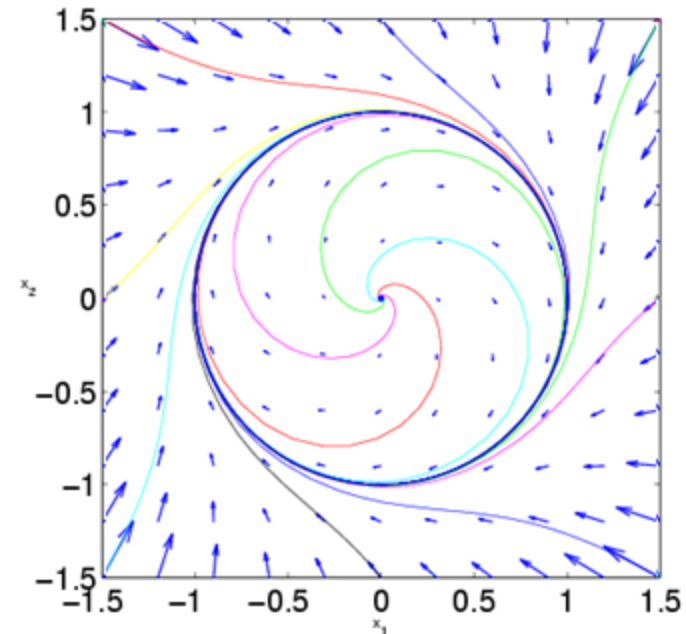
Example: Electronic Oscillator

- Outline
- Background
- Motivation
- First Result
- Next Idea
- Conclusions
- Future Work

- We want to study the structure of this system
- Linearize around limit cycle

$$\frac{dx_1}{dt} = x_2 + x_1 (1 - x_1^2 - x_2^2)$$

$$\frac{dx_2}{dt} = -x_1 + x_2 (1 - x_1^2 - x_2^2)$$

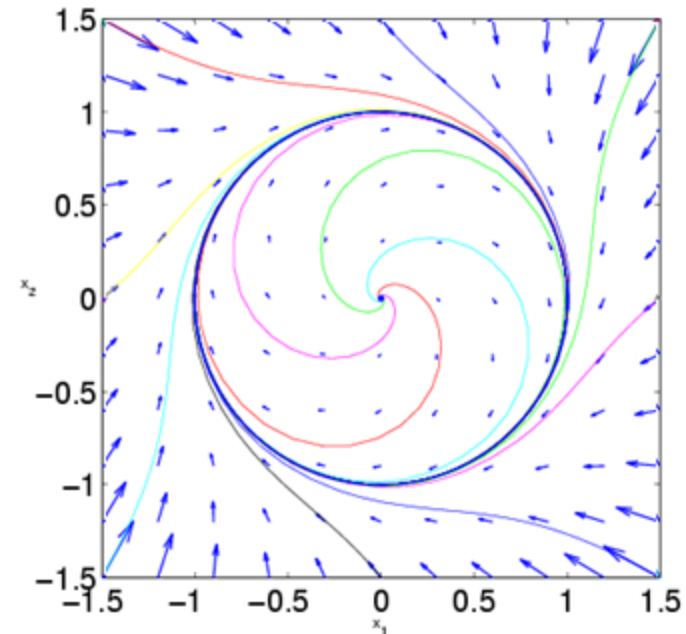


Example: Electronic Oscillator

- Outline
- Background
- Motivation
- First Result
- Next Idea
- Conclusions
- Future Work

- We want to study the structure of this system
- Linearize around limit cycle

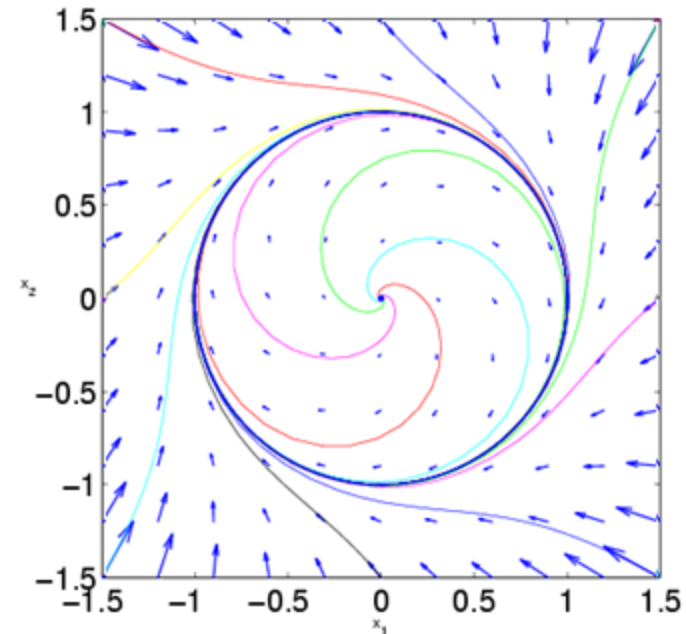
$$\begin{pmatrix} \dot{\delta}_1(t) \\ \dot{\delta}_2(t) \end{pmatrix} \approx \begin{pmatrix} -2\sin^2(t) & 1 - 2\sin(t)\cos(t) \\ 1 - 2\sin(t)\cos(t) & -2\cos^2(t) \end{pmatrix} \begin{pmatrix} \delta_1(t) \\ \delta_2(t) \end{pmatrix}$$



Example: Electronic Oscillator

- Outline
- Background
- Motivation
- First Result
- Next Idea
- Conclusions
- Future Work

- We want to study the structure of this system
- Linearize around limit cycle
- Need time invariant representation



$$\begin{pmatrix} \dot{\delta}_1(t) \\ \dot{\delta}_2(t) \end{pmatrix} \approx \begin{pmatrix} -2\sin^2(t) & 1 - 2\sin(t)\cos(t) \\ 1 - 2\sin(t)\cos(t) & -2\cos^2(t) \end{pmatrix} \begin{pmatrix} \delta_1(t) \\ \delta_2(t) \end{pmatrix}$$

First Results: Lifting

- This method finds a time-invariant representation of a time varying system.
- Maintains some properties of the system

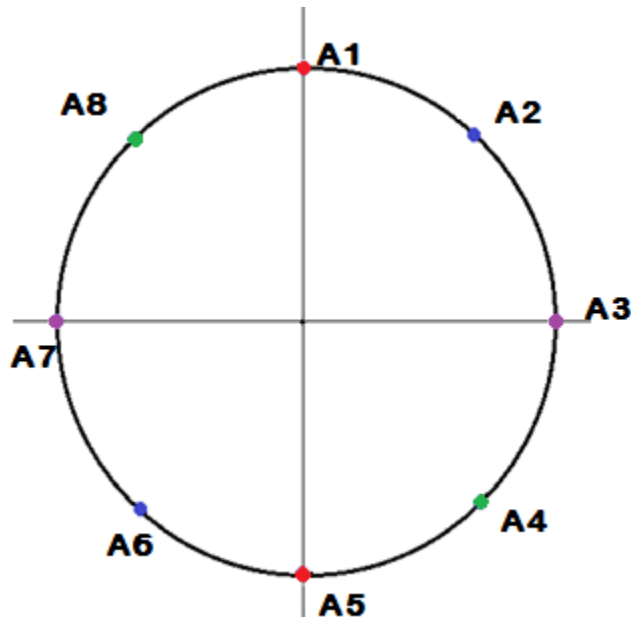
$$A_l = A(N-1)A(N-2)\dots A(0)$$
$$B_l = (A(N-1)A(N-2)\dots A(1)B(0) \quad A(N-1)\dots A(2)B(1) \quad \dots \quad B(N-1))$$
$$C_l = \begin{pmatrix} C(0) \\ C(1)A(0) \\ \vdots \\ C(N-1)A(N-2)\dots A(0) \end{pmatrix}$$

How to lift an N-periodic system.

Example

- Outline
- Background
- Motivation
- ▶ First Result
- Next Idea
- Conclusions
- Future Work

- Lifting the Oscillator



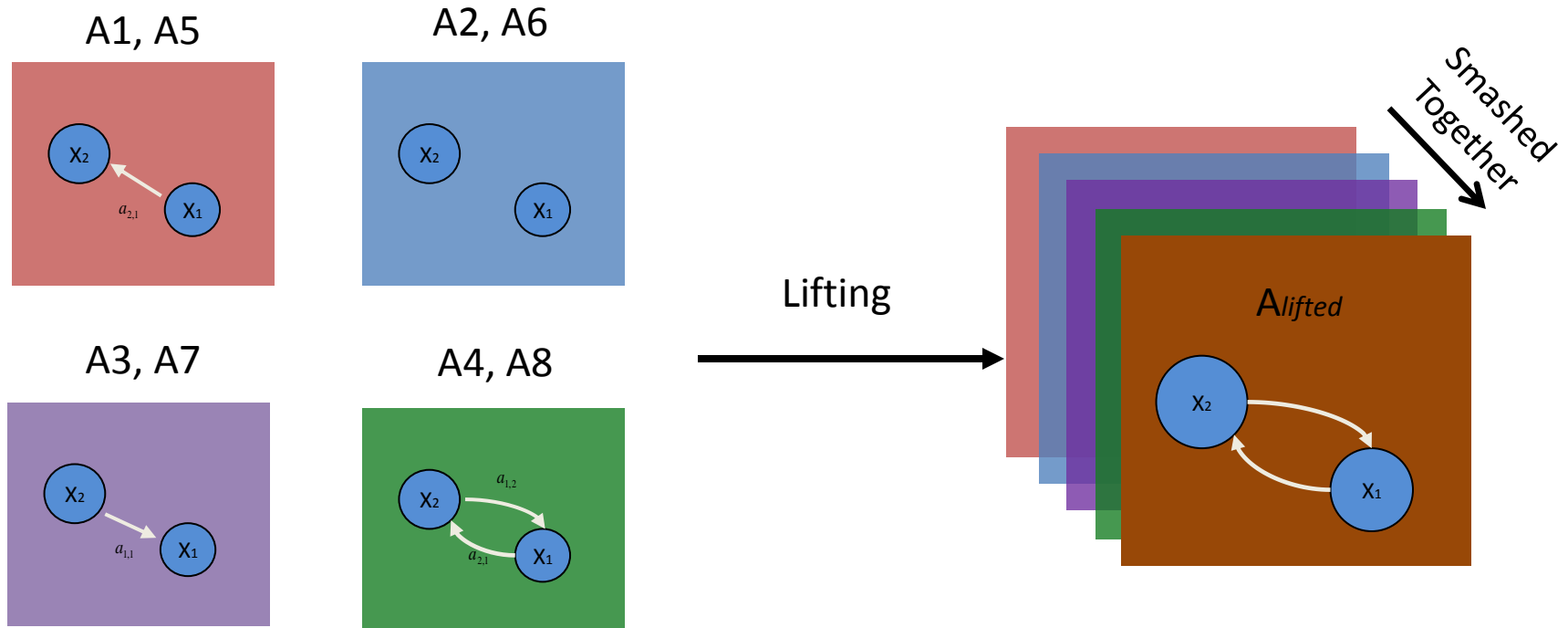
$$\begin{aligned} A1 &= \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} \\ A2 &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\ A3 &= \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} \\ A4 &= \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \\ A5 &= \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} \\ A6 &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\ A7 &= \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} \\ A8 &= \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \end{aligned}$$



$$A_l = \begin{pmatrix} 13 & -60 \\ -20 & 93 \end{pmatrix}$$

Lifting Does Not Preserve Structure!

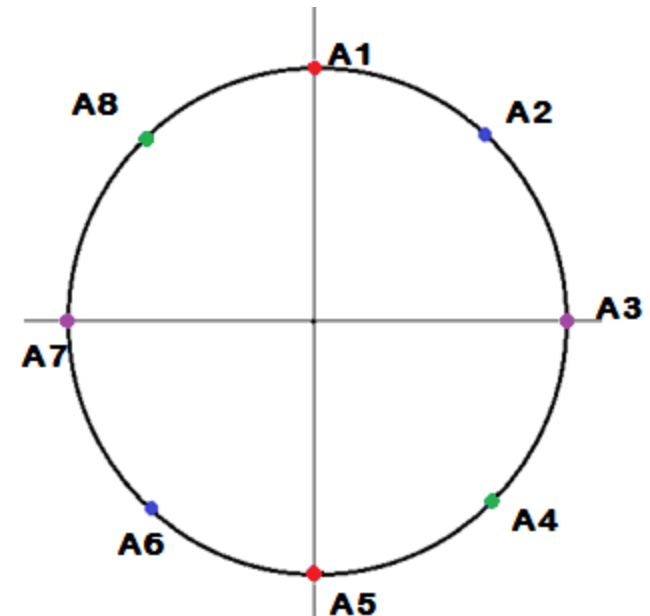
- Outline
- Background
- Motivation
- ▶ First Result
- Next Idea
- Conclusions
- Future Work



Next Idea: Decoupled Sampling

- Outline
- Background
- Motivation
- First Result
- ▶ Next Idea
- Conclusions
- Future Work

- Take samples of the system
- Find the DSF at each sample point
- Find a new time invariant representation
- Compute the DSF
- Compare



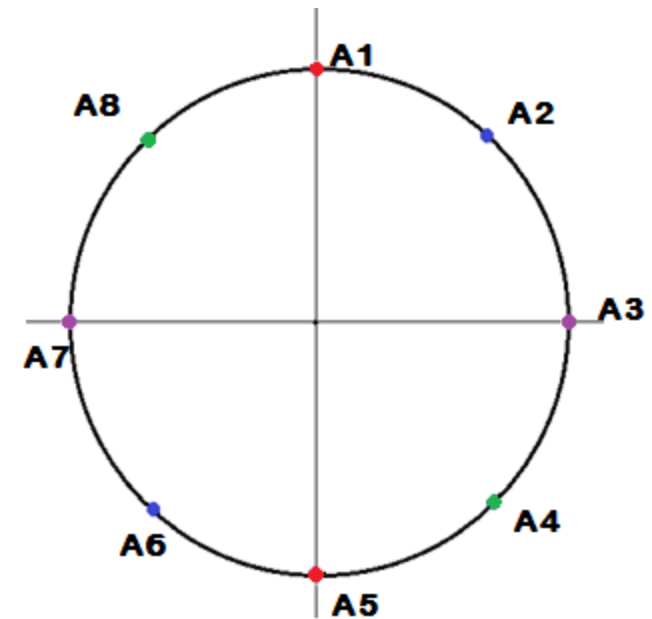
Decoupled Sampling

- Outline
- Background
- Motivation
- First Result
- ▶ Next Idea
- Conclusions
- Future Work

- How we obtain the time-invariant representation:

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \\ \dot{\bar{x}}_3 \\ \dot{\bar{x}}_4 \\ \dot{\bar{x}}_5 \\ \dot{\bar{x}}_6 \\ \dot{\bar{x}}_7 \\ \dot{\bar{x}}_8 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A7 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \\ \bar{x}_4 \\ \bar{x}_5 \\ \bar{x}_6 \\ \bar{x}_7 \\ \bar{x}_8 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \\ \bar{u}_4 \\ \bar{u}_5 \\ \bar{u}_6 \\ \bar{u}_7 \\ \bar{u}_8 \end{bmatrix}$$

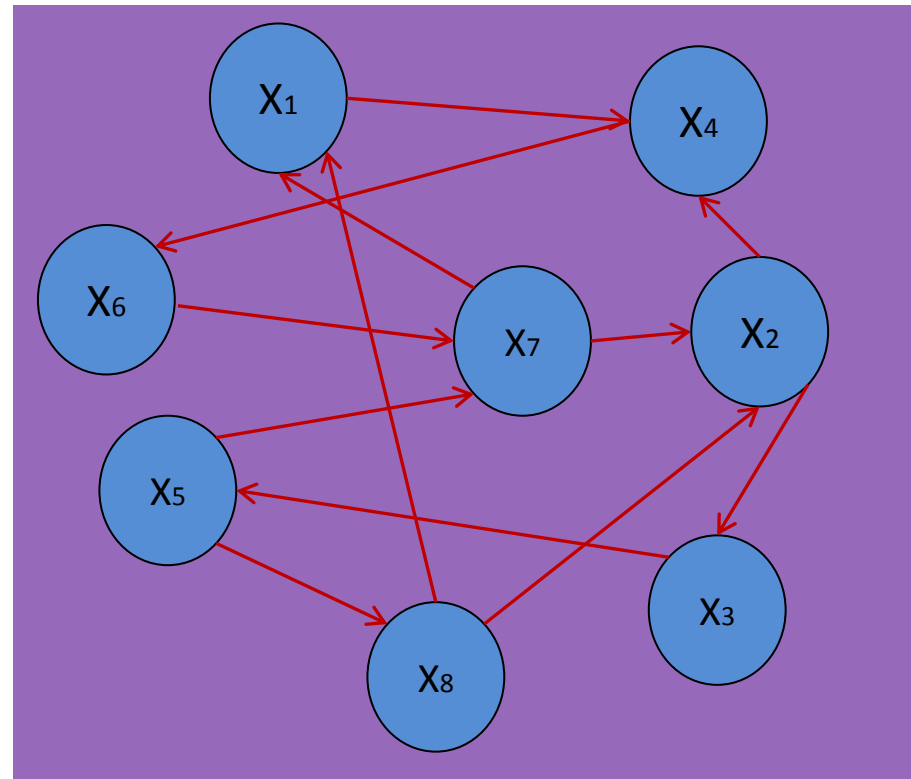
$$\begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \bar{y}_3 \\ \bar{y}_4 \\ \bar{y}_5 \\ \bar{y}_6 \\ \bar{y}_7 \\ \bar{y}_8 \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \\ \bar{x}_4 \\ \bar{x}_5 \\ \bar{x}_6 \\ \bar{x}_7 \\ \bar{x}_8 \end{bmatrix} \bar{H}$$



Decoupled Sampling: Results

- Outline
- Background
- Motivation
- First Result
- ▶ Next Idea
- Conclusions
- Future Work

- The structure of the system is kept
 - Contains information about the system



New A Matrix

Conclusions

- Outline
- Background
- Motivation
- First Result
- Next Idea
- ▶ Conclusions
- Future Work

- Structures of non linear systems with equilibrium points have been studied in the past
 - State Space Realization
 - Transfer Function
 - Dynamical Structure Function
- Limit cycles
- Developed a new way of representing time-varying systems with a time-invariant structure
 - Keeps structure

Future Work

- Outline
- Background
- Motivation
- First Result
- Next Idea
- Conclusions
- ▶ Future Work

- Developing Network Reconstruction algorithms to find the oscillating structure from data using the decoupled sampling technique
- Explore time varying representations of structure

References

Outline
Background
Motivation
First Result
Next Idea
Conclusions
Future Work

- [1] "Model reduction of periodic systems: a lifting approach," *Automatica*, vol. 41, no. 6, p. 1088, Jul. 2003.
- [2] "Analysis and Synthesis of Controllers for the Classes of Slowly Varying, Periodic and Multirate Systems," Ph.D Dissertation, Dept. AA, M.I.T., MA, 1991.
- [3] S. Warnick, "Dynamical Structure Functions for the Reverse Engineering of LTI Networks," *46th IEEE Conference on Decision and Control*, New Orleans, LA, 2007, pp. 1518-1519.