Necessary and Sufficient Conditions for Identifiability of Interconnected Subsystems

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Overview

- Partial System Representations
  - Interconnected Subsystems
  - Dynamical Structure Function

- Identifiability Conditions
  - Interconnected Subsystems
  - Dynamical Structure Function

- Comparison of Information Cost

- Numerical Example

- Conclusion
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Partial System Representations: Interconnected Subsystems

Classical representation of interconnected systems
Graphical representation called “subsystem structure”
Partial System Representations: Interconnected Subsystems

Denoted by the equations:
\[
\begin{bmatrix}
Y(s) \\
V(s)
\end{bmatrix}
= \begin{bmatrix}
N \\
W(s)
\end{bmatrix}
\begin{bmatrix}
U(s) \\
W(s)
\end{bmatrix}
\]

\[W(s) = S(s)V(s)\]

where

\[N = \begin{bmatrix}
0 & I \\
L & K
\end{bmatrix}\quad \text{and} \quad S(s) = \begin{bmatrix}
S_1(s) & 0 \\
0 & S_2(s)
\end{bmatrix}\]

Classical representation of interconnected systems

Graphical representation called “subsystem structure”
Partial System Representations: Interconnected Subsystems

Denoted by the equations:

\[
\begin{bmatrix}
Y(s) \\
V(s)
\end{bmatrix} = N \begin{bmatrix}
U(s) \\
W(s)
\end{bmatrix}
\]

\[W(s) = S(s)V(s)\]

where \(N = \begin{bmatrix}
0 & I \\
I & \bar{I}
\end{bmatrix}\) and \(S(s) = \begin{bmatrix}
G_1(s) & 0 \\
0 & G_2(s)
\end{bmatrix}\)

Classical representation of interconnected systems

Graphical representation called “subsystem structure”
Partial System Representations: Dynamical Structure Functions

Fluidic representation of interconnected systems
Graphical representation called “signal structure”
Partial System Representations: Dynamical Structure Functions

Denoted by the equation:
\[ Y(s) = Q(s)Y(s) + P(s)U(s) \]

Fluidic representation of interconnected systems
Graphical representation called “signal structure”
Partial System Representations: Dynamical Structure Functions

Denoted by the equation:

\[ Y(s) = Q(s)Y(s) + P(s)U(s) \]

For this example we have:

\[ Q(s) = \begin{bmatrix} 0 & G_1(s) \\ G_2(s) & 0 \end{bmatrix} \quad \text{and} \quad P(s) = \begin{bmatrix} G_1(s) & 0 \\ 0 & G_2(s) \end{bmatrix} \]

Fluidic representation of interconnected systems
Graphical representation called “signal structure”
Partial System Representations: Comparison of Representations

Appear to be graphical duals
(i.e. represent the same information)
Partial System Representations: Comparison of Representations

Different because dynamical structure functions can have shared state; States in interconnected systems cannot be shared across subsystems
Partial System Representations: Comparison of Representations

As different representations of a system, we pose the question:
which one is easier to reconstruct from data?
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Identifiability Conditions:
Interconnected Subsystems

\[ Y(s) = [I \ S(s)K]^{-1} S(s)LU(s) \]

\[ G(s) \]
Identifiability Conditions:
Interconnected Subsystems

\[ Y(s) = \left[ I \quad S(s)K \right]^{-1} S(s)LU(s) \]

\[ G(s) = \left[ I \quad S(s)K \right]^{-1} S(s)L \]
Identifiability Conditions: Interconnected Subsystems

\[ Y(s) = \left[ I \quad S(s)K \right]^{-1} S(s)LU(s) \]

\[ G(s) = \left[ I \quad S(s)K \right]^{-1} S(s)L \]

\[ G(s) = S(s)L + S(s)KG(s) \]
Identifiability Conditions: Interconnected Subsystems

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\[ G(s)^T = (S(s)L)^T + G(s)^T (S(s)K)^T \]
Identifiability Conditions: Interconnected Subsystems

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\[ \hat{g} = \begin{bmatrix} I \otimes I & I \otimes G(s)^T \end{bmatrix} \begin{bmatrix} \rightarrow s \ell \\ \rightarrow s k \end{bmatrix} \]
Identifiability Conditions: Interconnected Subsystems

\[ Y(s) = \begin{bmatrix} I & S(s)K \end{bmatrix}^{-1} S(s)LU(s) \]

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\[ \tilde{g} = \begin{bmatrix} I \otimes (I \otimes 1) & I \otimes (G(s)^T \otimes 1) \end{bmatrix} \]

\[ \tilde{g} = \begin{bmatrix} I \otimes I & I \otimes G(s)^T \end{bmatrix} \begin{bmatrix} \rightarrow \sl_1 \\ \vdots \\ \rightarrow \sl_{pmr} \\ \rightarrow \sk_1 \\ \vdots \\ \rightarrow \sk_{p^2r} \end{bmatrix} \]
Identifiability Conditions: Interconnected Subsystems

\[ Y(s) = [I \quad S(s)K]^{-1} S(s)LU(s) \]

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\[ G(s)^T = \begin{bmatrix} I & G(s)^T \\ \end{bmatrix} \begin{bmatrix} (S(s)L)^T \\ (S(s)K)^T \end{bmatrix} \]

Series of linear equations to solve for codependent parameters \( sl \) and \( sk \):

\[ \tilde{\mathbf{g}} = \begin{bmatrix} I \otimes (I \otimes 1) \quad I \otimes (G(s)^T \otimes 1) \end{bmatrix} \]

\[ \begin{bmatrix} sl_1 \\ \vdots \\ sl_{pmr} \\ sk_1 \\ \vdots \\ sk_{p^2r} \end{bmatrix} \]

\[ \begin{bmatrix} \rightarrow sl \\ \rightarrow sk \end{bmatrix} \]
Identifiability Conditions: Interconnected Subsystems

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\[ G(s) = S(s)L + S(s)KG(s) \]

\[ G(s)^T = (S(s)L)^T + G(s)^T (S(s)K)^T \]

\[ G(s)^T = \begin{bmatrix} I & G(s)^T \end{bmatrix} \begin{bmatrix} (S(s)L)^T \\ (S(s)K)^T \end{bmatrix} \]

Once we find \( sl \) and \( sk \), we can solve for \( s, l \) and \( k \) since \( l \) and \( k \) are boolean.

\[ \bar{g} = \begin{bmatrix} I \otimes (I \otimes 1) & I \otimes (G(s)^T \otimes 1) \end{bmatrix} \]

\[ \begin{bmatrix} \rightarrow \quad sl \quad \rightarrow \quad sk \end{bmatrix} \]
Identifiability Conditions: Interconnected Subsystems

**Information cost** is the number of parameters that must be known a priori to ensure a unique solution exists.

\[
\tilde{g} = \begin{bmatrix}
I \otimes (I \otimes 1) & I \otimes (G(s)^T \otimes 1)
\end{bmatrix}
\]

\[
\begin{align*}
&sl_1 \\
&\vdots \\
&sl_{pmr} \\
&sk_1 \\
&\vdots \\
&sk_{p^2r}
\end{align*}
\]
Identifiability Conditions: Interconnected Subsystems

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The higher the number of parameters known a priori, the more we can reduce the number of columns in this matrix.
Identifiability Conditions: Interconnected Subsystems

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Once it is full column rank, we can solve for the remaining parameters.
Identifiability Conditions: Interconnected Subsystems

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The higher the number of parameters known a priori, the more we can reduce the number of columns in this matrix.

Once it is full column rank, we can solve for the remaining parameters.

To potentially get it to full rank, we need to reduce the number of columns to $pm$ or less.
Identifiability Conditions: Interconnected Subsystems

Note that there are $pmr + p^2r$ parameters, where $r$ is the number of inputs into any subsystem.

\[
\bar{g} = \begin{bmatrix} I \otimes (I \otimes 1) & I \otimes (G(s)^T \otimes 1) \\
\end{bmatrix}
\]

\[
\begin{aligned}
sl_1 \\
\vdots \\
sl_{pmr} \\
\vdots \\
sk_1 \\
\vdots \\
sk_{p^2r}
\end{aligned}
\]
Identifiability Conditions: Interconnected Subsystems

Note that there are $pmr + p^2r$ parameters, where $r$ is the number of inputs into any subsystem.

\[
\bar{g} = \begin{bmatrix}
I \otimes (I \otimes 1) & I \otimes (G(s)^T \otimes 1)
\end{bmatrix}
\]

Without any information, we must assume that all subsystems can be connected to every other subsystem, which means we get

\[
r = p^2 - p + pm
\]
Identifiability Conditions: Interconnected Subsystems

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$$
\tilde{g} = \begin{bmatrix} 
I \otimes (I \otimes 1) & I \otimes (G(s)^T \otimes 1) 
\end{bmatrix}
$$

Without any information, we must assume that all subsystems can be connected to every other subsystem, which means we get

$$
r = p^2 - p + pm
$$

That means that the information cost for reconstructing interconnected subsystems is

$$
pm(p^2 - p + pm) + p^2(p^2 - p + pm) \quad pm
$$

$$
= p^4 + p^3 + 2p^3m + p^2m + p^2m^2 \quad pm
$$
Identifiability Conditions: Interconnected Subsystems

Note that there are $pmr + p^2r$ parameters, where $r$ is the number of inputs into any subsystem.

$$\bar{g} = \begin{bmatrix} I \otimes (I \otimes 1) & I \otimes (G(s)^T \otimes 1) \end{bmatrix}$$

Without any information, we must assume that all subsystems can be connected to every other subsystem, which means we get

$$r = p^2 - p + pm$$

If we make the additional assumption that all subsystems are single input, single output then we have $r = 1$ and the information cost becomes

$$pm + p^2 \quad pm = p^2$$
Identifiability Conditions: Dynamical Structure Functions

- Similar process for finding information cost of the dynamical structure function (results publish in CDC 2012 paper)

- Information cost is: $p^2 \cdot p$
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Comparison of Information Cost

Information cost of interconnected subsystems:

\[ p^4 + p^3 + 2p^3m + p^2m + p^2m^2 + pm \]

Information cost of dynamical structure functions:

\[ p^2 + p \]

Since \( p > 0 \) and \( m \geq 0 \):

\[ p^4 + p^3 + 2p^3m + p^2m + p^2m^2 + pm \]

\[ = p^2(p^2 + p) + pm(2p^2 + pm + p - 1) \]

\[ = (p^2 + p) + \text{where} \quad p > 0 \quad \text{and} \quad m \geq 0 \]

\[ (p^2 + p) + p^2 + p \]
Comparison of Information Cost

Information cost of SISO interconnected subsystems:

\[ p^2 \]

Information cost of dynamical structure functions:

\[ p^2 \quad p \]

Since \( p > 0 \) and \( m \neq 0 \):

\[ p^2 > p^2 \quad p \]
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Numerical Example

Assume we are given a transfer function: \( G(s) = \frac{1}{(s+1)(s+2)} \)
Numerical Example

Assume we are given a transfer function: 

\[ G(s) = \frac{1}{s+1} \frac{1}{(s+1)(s+2)} \]

We want to determine the unique interconnection of SISO subsystems that generated this transfer function, which is of the form

\[ N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ l_{11} & k_{11} & k_{12} \\ l_{21} & k_{21} & k_{22} \end{bmatrix} \]

and 

\[ S(s) = \begin{bmatrix} s_{11}(s) & 0 \\ 0 & s_{22}(s) \end{bmatrix} \]
Numerical Example

- The associated codependent parameters are \( x = \begin{bmatrix}
    s_{11}(s)l_{11} \\
    s_{22}(s)l_{22} \\
    s_{11}(s)k_{11} \\
    s_{11}(s)k_{12} \\
    s_{22}(s)k_{21} \\
    s_{22}(s)k_{22}
\end{bmatrix} \)
Numerical Example

- The associated codependent parameters are

\[
x = \begin{bmatrix}
  s_{11}(s)l_{11} \\
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  s_{11}(s)k_{12} \\
  s_{22}(s)k_{21} \\
  s_{22}(s)k_{22}
\end{bmatrix}
\]

- The information cost is \( p^2 = (2)^2 = 4 \), which means 4 codependent parameters (out of 6) must be known in order to reconstruct the interconnection of subsystems.
Numerical Example

- The associated codependent parameters are

\[ x = \begin{bmatrix}
    s_{11}(s)l_{11} \\
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    s_{11}(s)k_{11} \\
    s_{11}(s)k_{12} \\
    s_{22}(s)k_{21} \\
    s_{22}(s)k_{22}
\end{bmatrix} \]

- The information cost is \( p^2 = (2)^2 = 4 \), which means 4 codependent parameters (out of 6) must be known in order to reconstruct the interconnection of subsystems.

- If we assume that we know \( k_{11} = 0, k_{22} = 0, s_{22}(s)l_{22} = 0, \) and \( s_{11}(s)k_{12} = 0 \), then we get the equation:

\[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & \frac{1}{s+1}
\end{bmatrix}
\begin{bmatrix}
s_{11}(s)l_{11} \\
1 \\
s_{22}(s)k_{21}
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{s+1} \\
1 \\
\frac{1}{(s+1)(s+2)}
\end{bmatrix}
\]
Numerical Example

Solving the system of linear equations yields the interconnection of SISO subsystems:

\[
N = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
\end{bmatrix}
\]

and

\[
S(s) = \begin{bmatrix}
\frac{1}{s+1} & 0 \\
0 & \frac{1}{s+2} \\
\end{bmatrix}
\]
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- Cost of reconstructing interconnected subsystems is always greater than or equal to the cost of reconstructing the dynamical structure function.
Conclusion

- Cost of reconstructing interconnected subsystems is always greater than or equal to the cost of reconstructing the dynamical structure function.

- Even when we restrict ourselves to the class of SISO subsystems, which means that we know that the structure of $S(s)$ is diagonal, the cost of reconstructing the interconnected subsystems is higher than the cost of reconstructing the dynamical structure function.
The End
Questions?