+

Necessary and Sufficient Conditions for Identifiability of Interconnected Subsystems

Sean Warnick

Melbourne, Australia

Conference on Decision and Control 2017

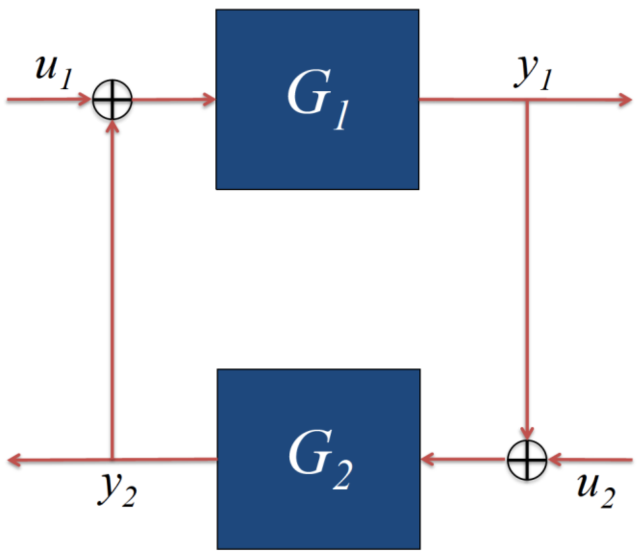
Overview

- Partial System Representations
  - Interconnected Subsystems
  - Dynamical Structure Function
- Identifiability Conditions
  - Interconnected Subsystems
  - Dynamical Structure Function
- Comparison of Information Cost
- Numerical Example
- Conclusion

+
Overview

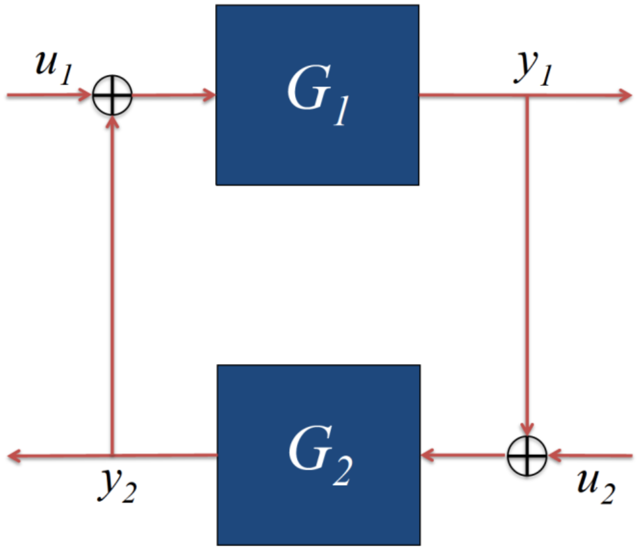
- Partial System Representations
  - Interconnected Subsystems
  - Dynamical Structure Function
- Identifiability Conditions
  - Interconnected Subsystems
  - Dynamical Structure Function
- Comparison of Information Cost
- Numerical Example
- Conclusion

Partial System Representations: Interconnected Subsystems



Classical representation of interconnected systems
Graphical representation called "subsystem structure"

Partial System Representations: Interconnected Subsystems



Denoted by the equations:

$$\begin{bmatrix} Y(s) \\ V(s) \end{bmatrix} = N \begin{bmatrix} U(s) \\ W(s) \end{bmatrix}$$

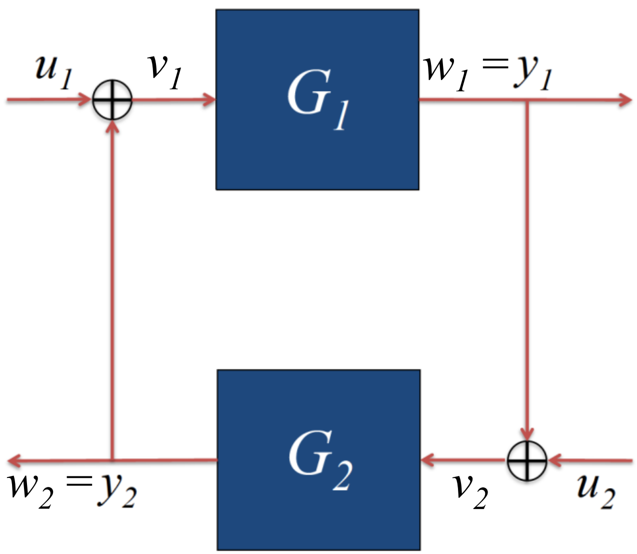
$$W(s) = S(s)V(s)$$

where

$N = \begin{bmatrix} 0 & I \\ L & K \end{bmatrix}$ and $S(s) = \begin{bmatrix} S_1(s) & 0 \\ 0 & S_2(s) \end{bmatrix}$

Classical representation of interconnected systems
Graphical representation called "subsystem structure"

Partial System Representations: Interconnected Subsystems



Denoted by the equations:

$$\begin{bmatrix} Y(s) \\ V(s) \end{bmatrix} = N \begin{bmatrix} U(s) \\ W(s) \end{bmatrix}$$

$$W(s) = S(s)V(s)$$

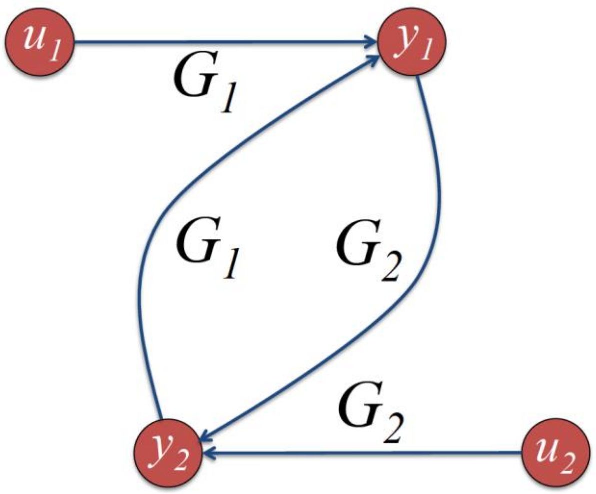
where

anti-diagonal

$$N = \begin{bmatrix} 0 & I \\ I & \overleftarrow{I} \end{bmatrix} \text{ and } S(s) = \begin{bmatrix} G_1(s) & 0 \\ 0 & G_2(s) \end{bmatrix}$$

Classical representation of interconnected systems
Graphical representation called "subsystem structure"

Partial System Representations: Dynamical Structure Functions

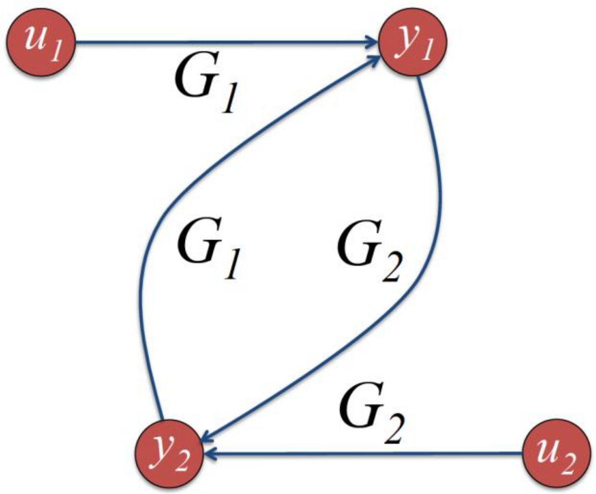


Fluidic representation of interconnected systems
Graphical representation called "signal structure"

Partial System Representations: Dynamical Structure Functions

Denoted by the equation:

$$Y(s) = Q(s)Y(s) + P(s)U(s)$$



Fluidic representation of interconnected systems
Graphical representation called "signal structure"

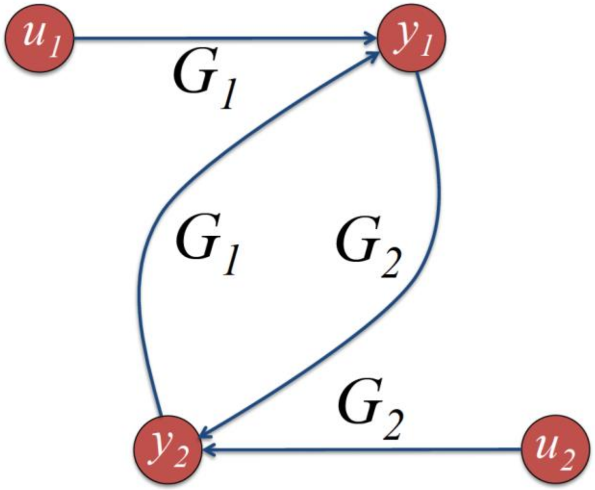
Partial System Representations: Dynamical Structure Functions

Denoted by the equation:

$$Y(s) = Q(s)Y(s) + P(s)U(s)$$

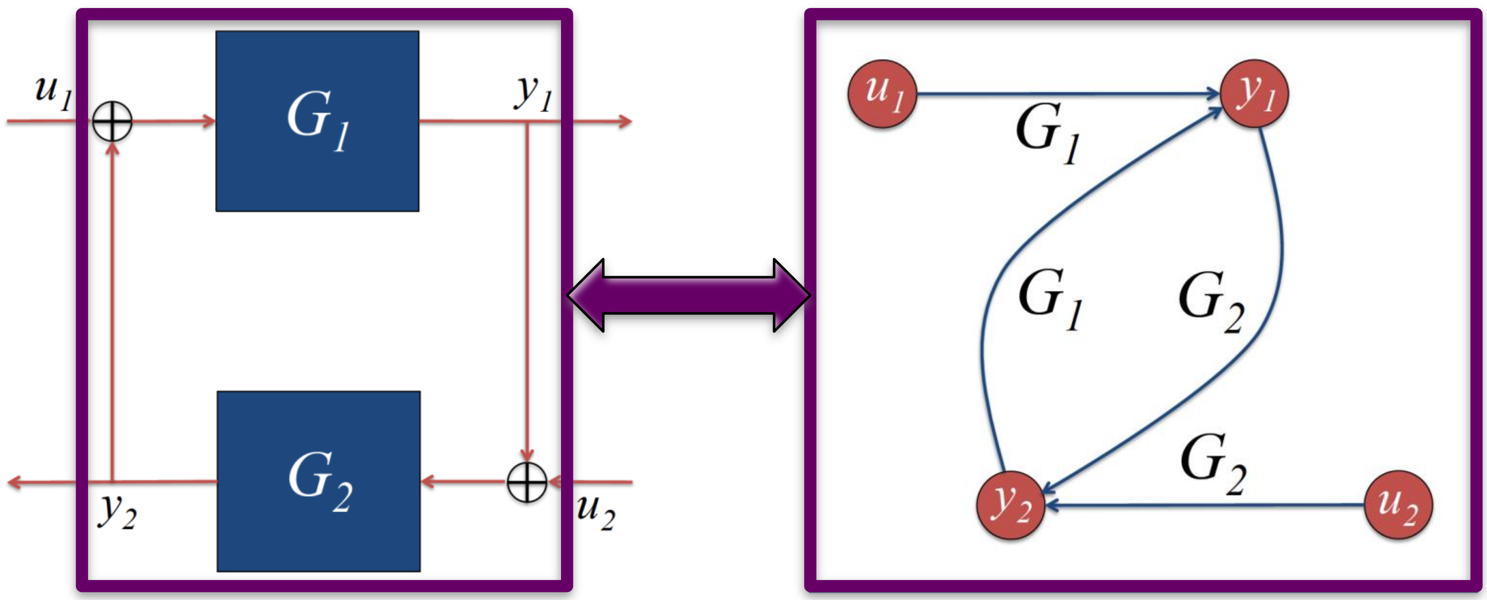
For this example we have:

$$Q(s) = \begin{bmatrix} 0 & G_1(s) \\ G_2(s) & 0 \end{bmatrix}$$ and $$P(s) = \begin{bmatrix} G_1(s) & 0 \\ 0 & G_2(s) \end{bmatrix}$$



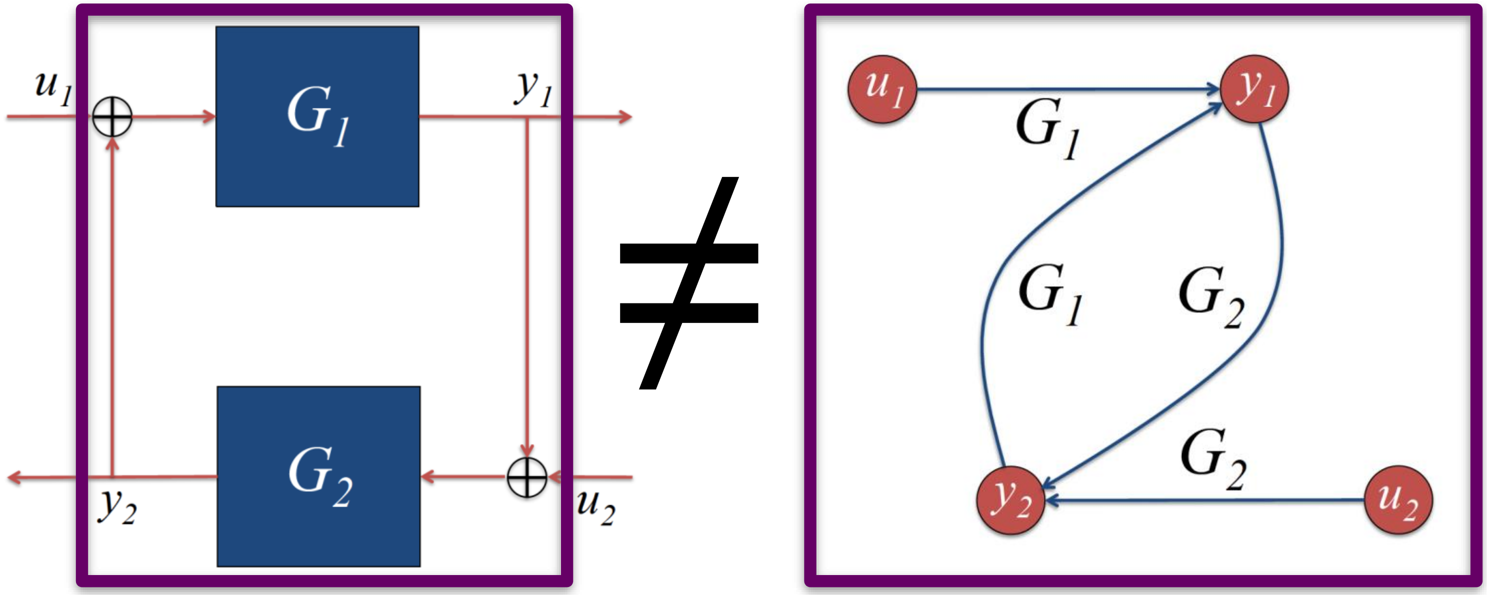
Fluidic representation of interconnected systems
Graphical representation called "signal structure"

Partial System Representations: Comparison of Representations



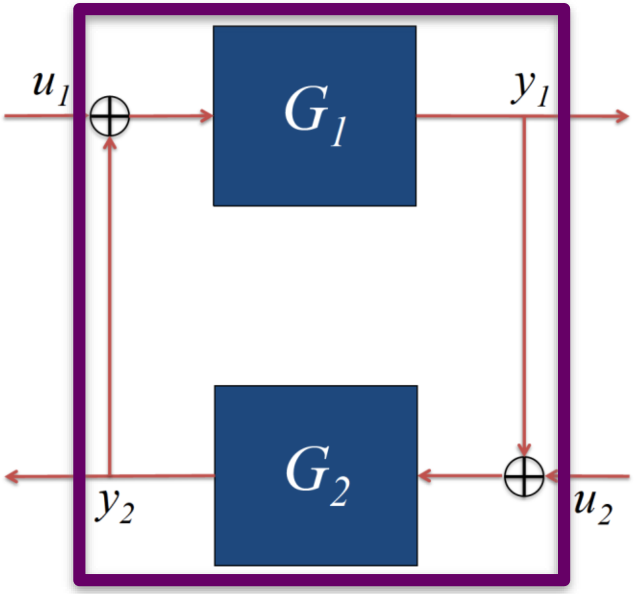
Appear to be graphical duals
(i.e. represent the same information)

Partial System Representations: Comparison of Representations

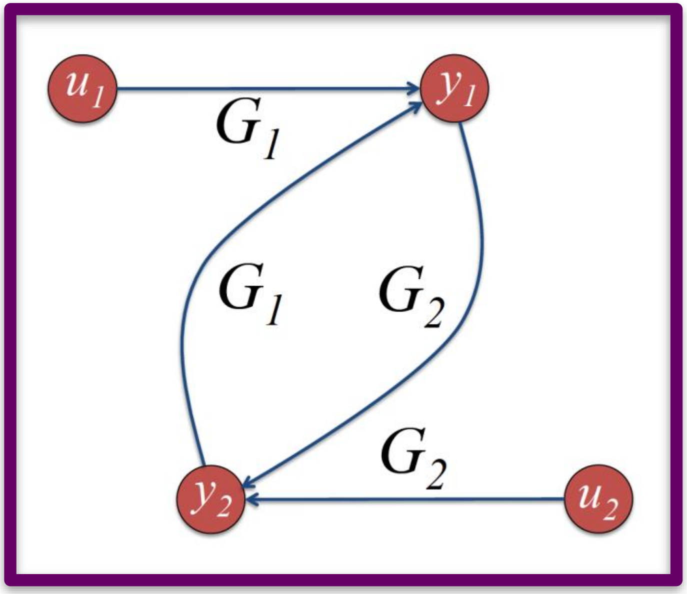


Different because dynamical structure functions can have shared state;
States in interconnected systems cannot be shared across subsystems

Partial System Representations: Comparison of Representations



?



As different representations of a system, we pose the question:
which one is easier to reconstruct from data?

Overview

- Partial System Representations
  - Interconnected Subsystems
  - Dynamical Structure Function
- Identifiability Conditions
  - Interconnected Subsystems
  - Dynamical Structure Function
- Comparison of Information Cost
- Numerical Example
- Conclusion

Identifiability Conditions: Interconnected Subsystems

$$Y(s) = \underbrace{[I - S(s)K]^{-1} S(s)L}_{G(s)} U(s)$$

Identifiability Conditions: Interconnected Subsystems

$$Y(s) = \underbrace{[I - S(s)K]^{-1} S(s) L}_{G(s)} U(s)$$

$$\Downarrow$$

$$G(s) = [I - S(s)K]^{-1} S(s) L$$

Identifiability Conditions: Interconnected Subsystems

$$Y(s) = \underbrace{[I - S(s)K]^{-1} S(s) L}_{G(s)} U(s)$$

$$\Downarrow$$

$$G(s) = [I - S(s)K]^{-1} S(s) L$$

$$\Downarrow$$

$$G(s) = S(s)L + S(s)KG(s)$$

Identifiability Conditions: Interconnected Subsystems

$$Y(s) = \underbrace{[I - S(s)K]^{-1} S(s)L}_{G(s)} U(s)$$

$$\Downarrow$$

$$G(s) = [I - S(s)K]^{-1} S(s)L$$

$$\Downarrow$$

$$G(s) = S(s)L + S(s)KG(s)$$

$$\Downarrow$$

$$G(s)^T = (S(s)L)^T + G(s)^T (S(s)K)^T$$

Identifiability Conditions: Interconnected Subsystems

$$Y(s) = [I - S(s)K]^{-1} S(s) L U(s)$$

$$\underbrace{[I - S(s)K]^{-1} S(s) L}_{G(s)}$$

$$G(s) = [I - S(s)K]^{-1} S(s) L$$

$$G(s) = S(s)L + S(s)KG(s)$$

$$G(s)^T = (S(s)L)^T + G(s)^T (S(s)K)^T$$

$$G(s)^T = \begin{bmatrix} I & G(s)^T \end{bmatrix} \begin{bmatrix} (S(s)L)^T \\ (S(s)K)^T \end{bmatrix}$$

Identifiability Conditions: Interconnected Subsystems

$$Y(s) = \underbrace{[I - S(s)K]^{-1} S(s)L}_{G(s)} U(s)$$

$$G(s) = [I - S(s)K]^{-1} S(s)L$$

$$G(s) = S(s)L + S(s)KG(s)$$

$$G(s)^T = (S(s)L)^T + G(s)^T (S(s)K)^T$$

$$G(s)^T = \begin{bmatrix} I & G(s)^T \end{bmatrix} \begin{bmatrix} (S(s)L)^T \\ (S(s)K)^T \end{bmatrix}$$

$$\vec{g} = \begin{bmatrix} I \otimes I & I \otimes G(s)^T \end{bmatrix} \begin{bmatrix} \vec{sl} \\ \vec{sk} \end{bmatrix}$$

Identifiability Conditions: Interconnected Subsystems

$$Y(s) = \underbrace{[I - S(s)K]^{-1}S(s)L}_{G(s)}U(s)$$

$$G(s) = [I - S(s)K]^{-1}S(s)L$$

$$G(s) = S(s)L + S(s)KG(s)$$

$$G(s)^T = (S(s)L)^T + G(s)^T(S(s)K)^T$$

$$G(s)^T = \begin{bmatrix} I & G(s)^T \end{bmatrix} \begin{bmatrix} (S(s)L)^T \\ (S(s)K)^T \end{bmatrix}$$

$$\vec{g} = \begin{bmatrix} I \otimes I & I \otimes G(s)^T \end{bmatrix} \begin{bmatrix} \vec{sl} \\ \vec{sk} \end{bmatrix}$$

$$\vec{g} = \begin{bmatrix} I \otimes (I \otimes 1) & I \otimes (G(s)^T \otimes 1) \end{bmatrix} \begin{bmatrix} sl_1 \\ \vdots \\ sl_{pmr} \\ sk_1 \\ \vdots \\ sk_{p^2r} \end{bmatrix}$$

Identifiability Conditions: Interconnected Subsystems

$$Y(s) = [I - S(s)K]^{-1} S(s) L U(s)$$

$$G(s)$$

$$G(s) = [I - S(s)K]^{-1} S(s) L$$

$$G(s) = S(s)L + S(s)KG(s)$$

$$G(s)^T = (S(s)L)^T + G(s)^T (S(s)K)^T$$

$$G(s)^T = \begin{bmatrix} I & G(s)^T \end{bmatrix} \begin{bmatrix} (S(s)L)^T \\ (S(s)K)^T \end{bmatrix}$$

$$\vec{g} = \begin{bmatrix} I \otimes I & I \otimes G(s)^T \end{bmatrix} \begin{bmatrix} \vec{sl} \\ \vec{sk} \end{bmatrix}$$

Series of linear equations to solve for codependent parameters ***sl*** and ***sk*** :

$$\vec{g} = \begin{bmatrix} I \otimes (I \otimes 1) & I \otimes (G(s)^T \otimes 1) \end{bmatrix} \begin{bmatrix} sl_1 \\ \vdots \\ sl_{pmr} \\ sk_1 \\ \vdots \\ sk_{p^2 r} \end{bmatrix}$$

Identifiability Conditions: Interconnected Subsystems

$$Y(s) = \underbrace{[I - S(s)K]^{-1} S(s)L}_{G(s)} U(s)$$

$$G(s) = [I - S(s)K]^{-1} S(s)L$$

$$G(s) = S(s)L + S(s)KG(s)$$

$$G(s)^T = (S(s)L)^T + G(s)^T (S(s)K)^T$$

$$G(s)^T = \begin{bmatrix} I & G(s)^T \end{bmatrix} \begin{bmatrix} (S(s)L)^T \\ (S(s)K)^T \end{bmatrix}$$

$$\vec{g} = \begin{bmatrix} I \otimes I & I \otimes G(s)^T \end{bmatrix} \begin{bmatrix} \vec{sl} \\ \vec{sk} \end{bmatrix}$$

$$\vec{g} = \begin{bmatrix} I \otimes (I \otimes 1) & I \otimes (G(s)^T \otimes 1) \end{bmatrix} \begin{bmatrix} sl_1 \\ \vdots \\ sl_{pmr} \\ sk_1 \\ \vdots \\ sk_{p^2 r} \end{bmatrix}$$

Once we find ***sl*** and ***sk***, we can solve for ***s***, ***l*** and ***k*** since ***l*** and ***k*** are boolean

Identifiability Conditions: Interconnected Subsystems

Information cost is the number of parameters that must be known a priori to ensure a unique solution exists

$$\vec{g} = \begin{bmatrix} I \otimes (I \otimes 1) & I \otimes (G(s)^T \otimes 1) \end{bmatrix} \begin{bmatrix} sl_1 \\ \vdots \\ sl_{pmr} \\ sk_1 \\ \vdots \\ sk_{p^2r} \end{bmatrix}$$

Identifiability Conditions: Interconnected Subsystems

Information cost is the number of parameters that must be known a priori to ensure a unique solution exists

$$\vec{g} = \begin{bmatrix} I \otimes (I \otimes 1) & I \otimes (G(s)^T \otimes 1) \end{bmatrix} \begin{bmatrix} sl_1 \\ \vdots \\ sl_{pmr} \\ sk_1 \\ \vdots \\ sk_{p^2r} \end{bmatrix}$$

The higher the number of parameters known a priori, the more we can reduce the number of columns in this matrix.

Identifiability Conditions: Interconnected Subsystems

Information cost is the number of parameters that must be known a priori to ensure a unique solution exists

$$\vec{g} = \begin{bmatrix} I \otimes (I \otimes 1) & I \otimes (G(s)^T \otimes 1) \end{bmatrix} \begin{bmatrix} sl_1 \\ \vdots \\ sl_{pmr} \\ sk_1 \\ \vdots \\ sk_{p^2 r} \end{bmatrix}$$

The higher the number of parameters known a priori, the more we can reduce the number of columns in this matrix.

Once it is full column rank, we can solve for the remaining parameters.

Identifiability Conditions: Interconnected Subsystems

Information cost is the number of parameters that must be known a priori to ensure a unique solution exists

$$\vec{g} = \begin{bmatrix} I \otimes (I \otimes 1) & I \otimes (G(s)^T \otimes 1) \end{bmatrix} \begin{bmatrix} sl_1 \\ \vdots \\ sl_{pmr} \\ sk_1 \\ \vdots \\ sk_{p^2 r} \end{bmatrix}$$

The higher the number of parameters known a priori, the more we can reduce the number of columns in this matrix.

Once it is full column rank, we can solve for the remaining parameters.

To potentially get it to full rank, we need to reduce the number of columns to pm or less.

Identifiability Conditions: Interconnected Subsystems

Note that there are $pmr + p^2r$ parameters, where r is the number of inputs into any subsystem.

$$\vec{g} = \begin{bmatrix} I \otimes (I \otimes 1) & I \otimes (G(s)^T \otimes 1) \end{bmatrix} \begin{bmatrix} sl_1 \\ \vdots \\ sl_{pmr} \\ sk_1 \\ \vdots \\ sk_{p^2r} \end{bmatrix}$$

Identifiability Conditions: Interconnected Subsystems

Note that there are $pmr + p^2r$ parameters, where r is the number of inputs into any subsystem.

$$\vec{g} = \begin{bmatrix} I \otimes (I \otimes 1) & I \otimes (G(s)^T \otimes 1) \end{bmatrix} \begin{bmatrix} sl_1 \\ \vdots \\ sl_{pmr} \\ sk_1 \\ \vdots \\ sk_{p^2r} \end{bmatrix}$$

Without any information, we must assume that all subsystems can be connected to every other subsystem, which means we get

$$r = p^2 - p + pm$$

Identifiability Conditions: Interconnected Subsystems

Note that there are $pmr + p^2r$ parameters, where r is the number of inputs into any subsystem.

$$\vec{g} = \begin{bmatrix} I \otimes (I \otimes 1) & I \otimes (G(s)^T \otimes 1) \end{bmatrix} \begin{bmatrix} sl_1 \\ \vdots \\ sl_{pmr} \\ sk_1 \\ \vdots \\ sk_{p^2r} \end{bmatrix}$$

Without any information, we must assume that all subsystems can be connected to every other subsystem, which means we get

$$r = p^2 - p + pm$$

That means that the information cost for reconstructing interconnected subsystems is

$$pm(p^2 - p + pm) + p^2(p^2 - p + pm) - pm$$
$$= p^4 - p^3 + 2p^3m - p^2m + p^2m^2 - pm$$

Identifiability Conditions: Interconnected Subsystems

Note that there are $pmr + p^2r$ parameters, where r is the number of inputs into any subsystem.

$$\vec{g} = \begin{bmatrix} I \otimes (I \otimes 1) & I \otimes (G(s)^T \otimes 1) \end{bmatrix} \begin{bmatrix} sl_1 \\ \vdots \\ sl_{pmr} \\ sk_1 \\ \vdots \\ sk_{p^2r} \end{bmatrix}$$

Without any information, we must assume that all subsystems can be connected to every other subsystem, which means we get

$$r = p^2 - p + pm$$

If me make the additional assumption that all subsystems are single input, single output then we have $r = 1$ and the information cost becomes

$$pm + p^2 - pm = p^2$$

Identifiability Conditions: Dynamical Structure Functions

- Similar process for finding information cost of the dynamical structure function (results publish in CDC 2012 paper)
- **Information cost** is: $p^2 - p$

+ Overview

- Partial System Representations
  - Interconnected Subsystems
  - Dynamical Structure Function
- Identifiability Conditions
  - Interconnected Subsystems
  - Dynamical Structure Function
- Comparison of Information Cost
- Numerical Example
- Conclusion

Comparison of Information Cost

Information cost of interconnected subsystems:

$$p^4 - p^3 + 2p^3m - p^2m + p^2m^2 - pm$$

Information cost of dynamical structure functions:

$$p^2 - p$$

Since $p > 0$ and $m \geq 0$: $p^4 - p^3 + 2p^3m - p^2m + p^2m^2 - pm$

$$= p^2(p^2 - p) + pm(2p^2 + pm - p - 1)$$

$$= a(p^2 - p) + b \quad \text{where } a > 0 \text{ and } b \geq 0$$

$$\therefore \; a(p^2 - p) + b \geq p^2 - p$$

Comparison of Information Cost

Information cost of SISO interconnected subsystems:

$$p^2$$

Information cost of dynamical structure functions:

$$p^2 - p$$

Since $p > 0$ and $m \geq 0$:

$$p^2 > p^2 - p$$

Overview

- Partial System Representations
  - Interconnected Subsystems
  - Dynamical Structure Function
- Identifiability Conditions
  - Interconnected Subsystems
  - Dynamical Structure Function
- Comparison of Information Cost
- Numerical Example
- Conclusion

Numerical Example

- Assume we are given a transfer function: $G(s) = \begin{bmatrix} \dfrac{1}{s+1} \\ \dfrac{1}{(s+1)(s+2)} \end{bmatrix}$

+

Numerical Example

- Assume we are given a transfer function: $G(s)=\begin{bmatrix} \dfrac{1}{s+1} \\ \dfrac{1}{(s+1)(s+2)} \end{bmatrix}$

- We want to determine the unique interconnection of SISO subsystems that generated this transfer function, which is of the form

$$N=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ l_{11} & k_{11} & k_{12} \\ l_{21} & k_{21} & k_{22} \end{bmatrix} \quad \text{and} \quad S(s)=\begin{bmatrix} s_{11}(s) & 0 \\ 0 & s_{22}(s) \end{bmatrix}$$

Numerical Example

- The associated codependent parameters are $x = \begin{bmatrix} s_{11}(s)l_{11} \\ s_{22}(s)l_{22} \\ s_{11}(s)k_{11} \\ s_{11}(s)k_{12} \\ s_{22}(s)k_{21} \\ s_{22}(s)k_{22} \end{bmatrix}$

Numerical Example

- The associated codependent parameters are $x = \begin{bmatrix} s_{11}(s)l_{11} \\ s_{22}(s)l_{22} \\ s_{11}(s)k_{11} \\ s_{11}(s)k_{12} \\ s_{22}(s)k_{21} \\ s_{22}(s)k_{22} \end{bmatrix}$

- The information cost is $p^2 = (2)^2 = 4$, which means 4 codependent parameters (out of 6) must be known in order to reconstruct the interconnection of subsystems

Numerical Example

- The associated codependent parameters are $x = \begin{bmatrix} s_{11}(s)l_{11} \\ s_{22}(s)l_{22} \\ s_{11}(s)k_{11} \\ s_{11}(s)k_{12} \\ s_{22}(s)k_{21} \\ s_{22}(s)k_{22} \end{bmatrix}$
- The information cost is $p^2 = (2)^2 = 4$, which means 4 codependent parameters (out of 6) must be known in order to reconstruct the interconnection of subsystems
- If we assume that we know $k_{11} = 0$, $k_{22} = 0$, $s_{22}(s)l_{22} = 0$, and $s_{11}(s)k_{12} = 0$, then we get the equation:

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} s_{11}(s)l_{11} \\ s_{22}(s)k_{21} \end{bmatrix} = \begin{bmatrix} \frac{1}{s+1} \\ \frac{1}{(s+1)(s+2)} \end{bmatrix}$$

Numerical Example

- Solving the system of linear equations yields the interconnection of SISO subsystems:

$$N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad S(s) = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+2} \end{bmatrix}$$

Overview

- Partial System Representations
  - Interconnected Subsystems
  - Dynamical Structure Function
- Identifiability Conditions
  - Interconnected Subsystems
  - Dynamical Structure Function
- Comparison of Information Cost
- Numerical Example
- Conclusion

Conclusion

- Cost of reconstructing interconnected subsystems is **always greater than or equal to** the cost of reconstructing the dynamical structure function

Conclusion

- Cost of reconstructing interconnected subsystems is **always greater than or equal to** the cost of reconstructing the dynamical structure function

- Even when we restrict the ourselves to the class of SISO subsystems, which means that **we know that the structure of $S(s)$ is diagonal**, the cost of reconstructing the interconnected subsystems is **higher than the cost** of reconstructing the dynamical structure function

The End

Questions?