Dynamic Networks: Representations, Abstractions, and Well-Posedness

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Outline

Introduction

- Dynamical Networks, their Signal Structures, and the Dynamical Structure Function (DSF)

Abstractions of DSFs

- Definition and Computation

Well-Posedness

- Necessary and Sufficient Conditions for the Well-Posedness of DSFs and their Abstractions

Network Reconstruction

- Motivation for Abstractions

Introduction

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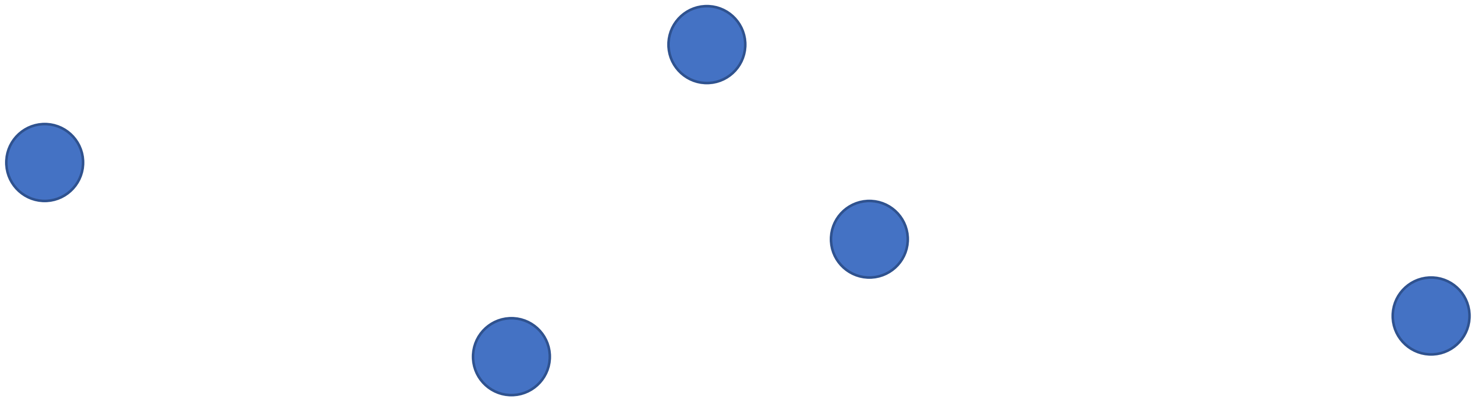
  - Nodes are manifest variables



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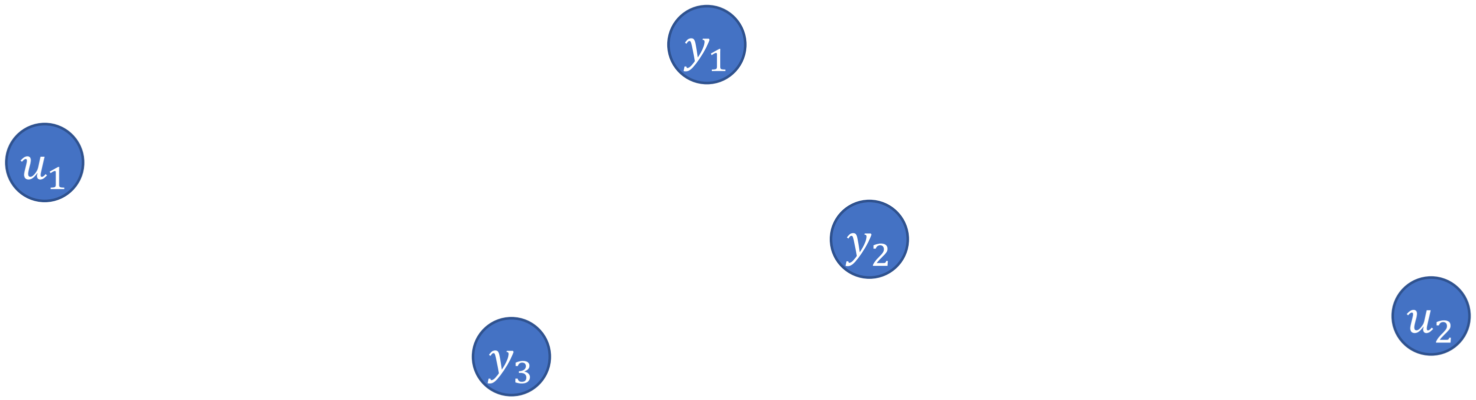

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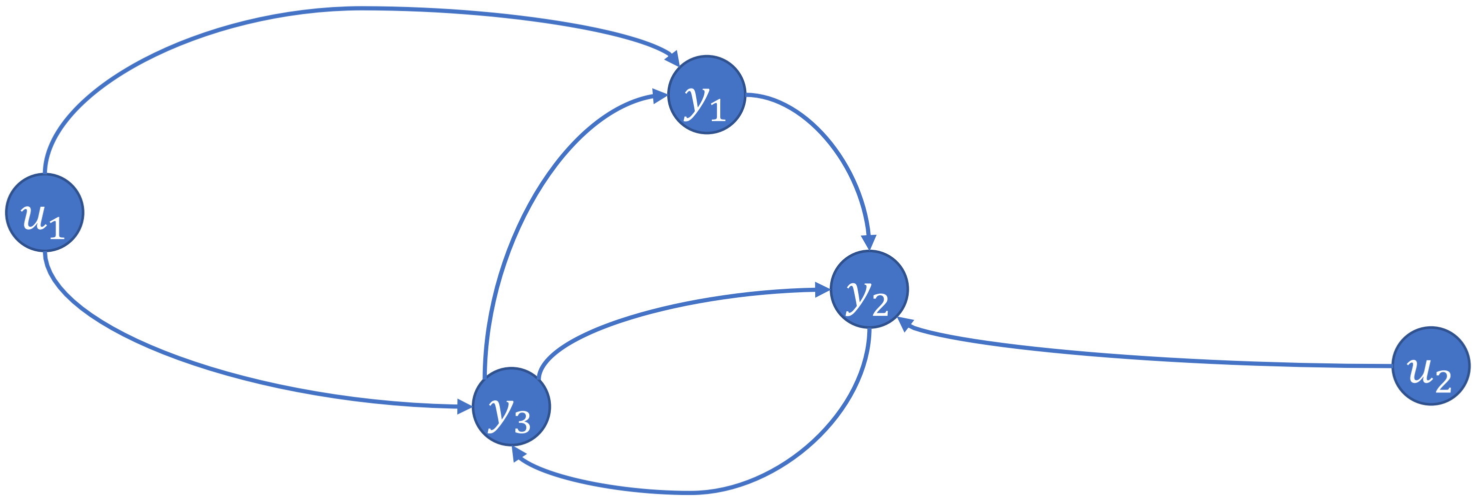

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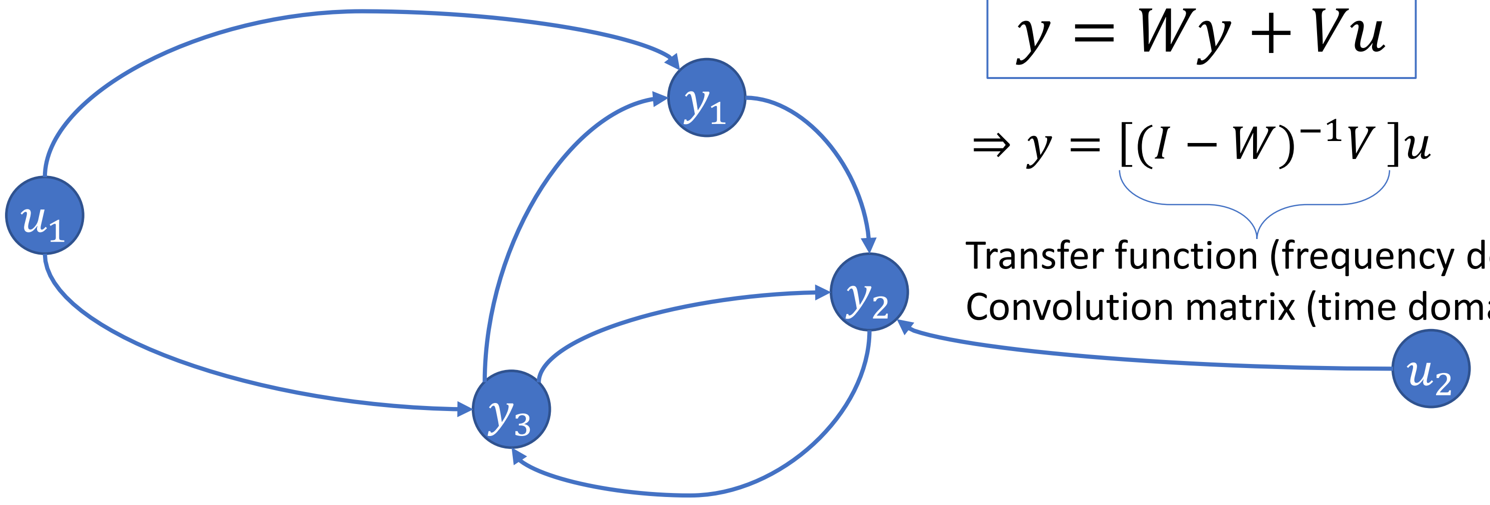
  - Nodes are manifest variables
    - Partitioned into inputs u and outputs y (without loss of generality, scalar valued)
  - Edges are causal dependencies (proper rational functions for LTI systems)
  - LTI signal structure represented by a Dynamical Network Function (DNF)



$$y = Wy + Vu$$

$$\Rightarrow y = \underbrace{[(I - W)^{-1} V\,]}u$$

Transfer function (frequency domain) or
Convolution matrix (time domain)

Dynamical Structure Function (DSF)

- LTI signal structure (DNF) with no self loops (Q(s) is zero along its diagonal)

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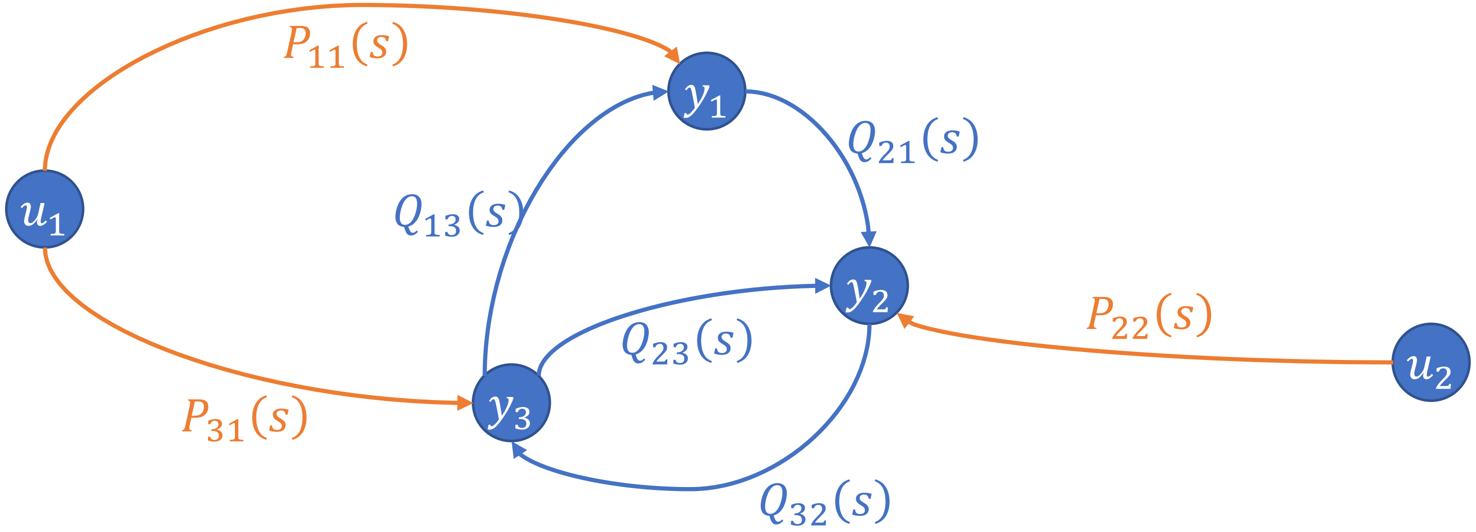
- Example:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & Q_{13}(s) \\ Q_{21}(s) & 0 & Q_{23}(s) \\ 0 & Q_{32}(s) & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} P_{11}(s) & 0 \\ 0 & P_{22}(s) \\ P_{31}(s) & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$



Abstractions of DSFs

Abstractions Definition

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- Let S be the nodes we keep measuring, $\bar{S}$ the nodes we abstract away
- An abstraction of (Q, P) is another DSF (Q_S, P_S) such that
  - The input-output behavior is preserved
  - The causal dependencies between the remaining manifest variables is preserved

Computing the Abstraction

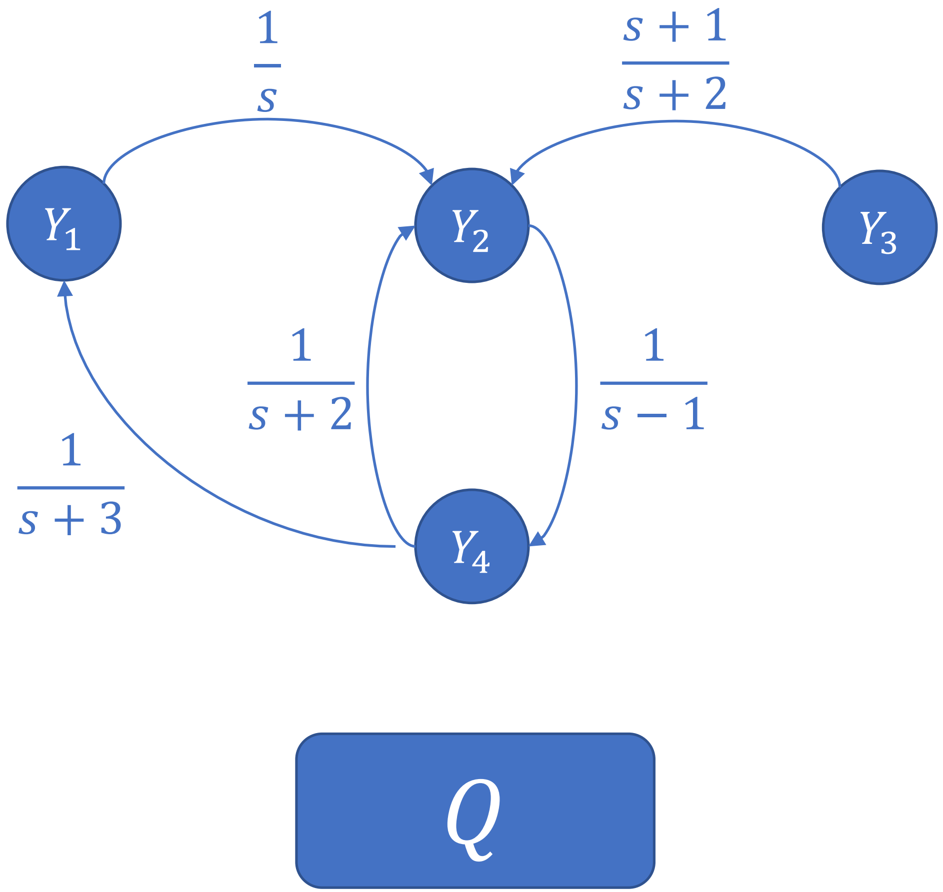
Realization (Q, P)

- $\begin{bmatrix} Y_S \\ Y_{\bar{S}} \end{bmatrix} = \underbrace{\begin{bmatrix} Q_{SS} & Q_{S\bar{S}} \\ Q_{\bar{S}S} & Q_{\bar{S}\bar{S}} \end{bmatrix}}_{Q} \begin{bmatrix} Y_S \\ Y_{\bar{S}} \end{bmatrix} + \underbrace{\begin{bmatrix} P_{SS} \\ P_{\bar{S}\bar{S}} \end{bmatrix}}_{P} U$

Abstraction (Q_S, P_S)

- $W = Q_{SS} + Q_{S\bar{S}}(I - Q_{\bar{S}\bar{S}})^{-1} Q_{\bar{S}S}$
- $V = P_{SS} + Q_{S\bar{S}}(I - Q_{\bar{S}\bar{S}})^{-1} P_{\bar{S}\bar{S}}$
- $D_W = diag(W)$
- $Q_S = (I - D_W)^{-1}(W - D_W)$
- $P_S = (I - D_W)^{-1} V$
- $\boldsymbol{Y_S = Q_S Y_S + P_S U}$

Abstraction: Example



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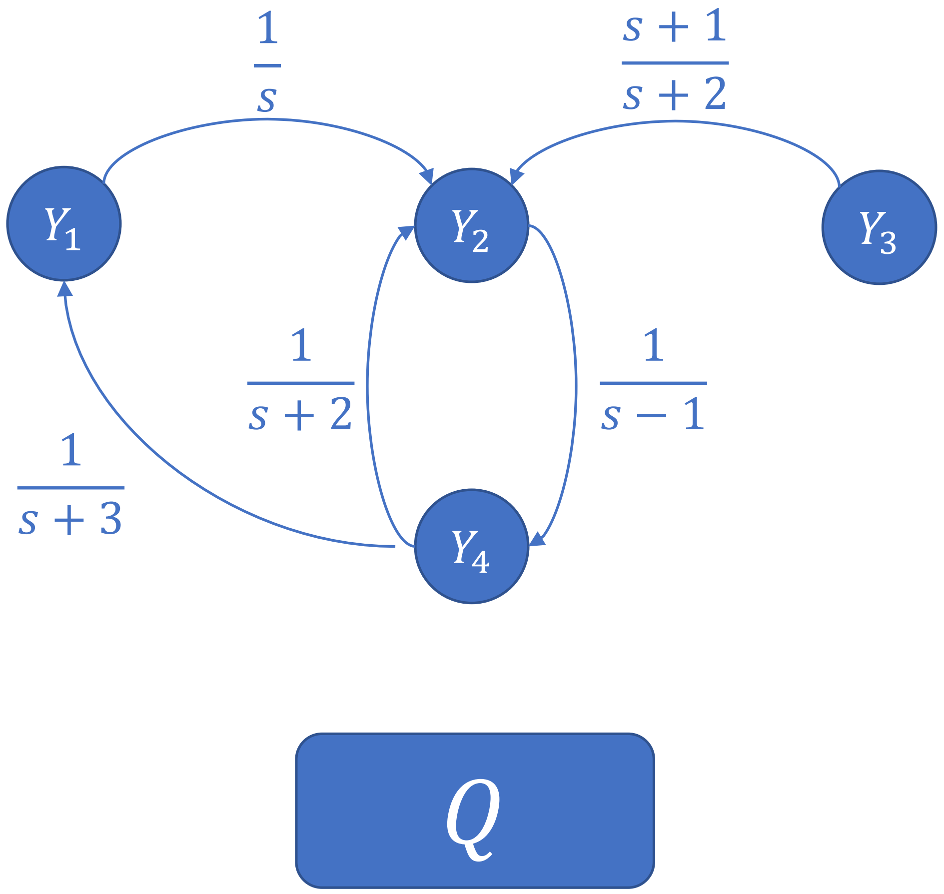

Let $S = \{1,2,3\}$
$\bar{S} = \{4\}$

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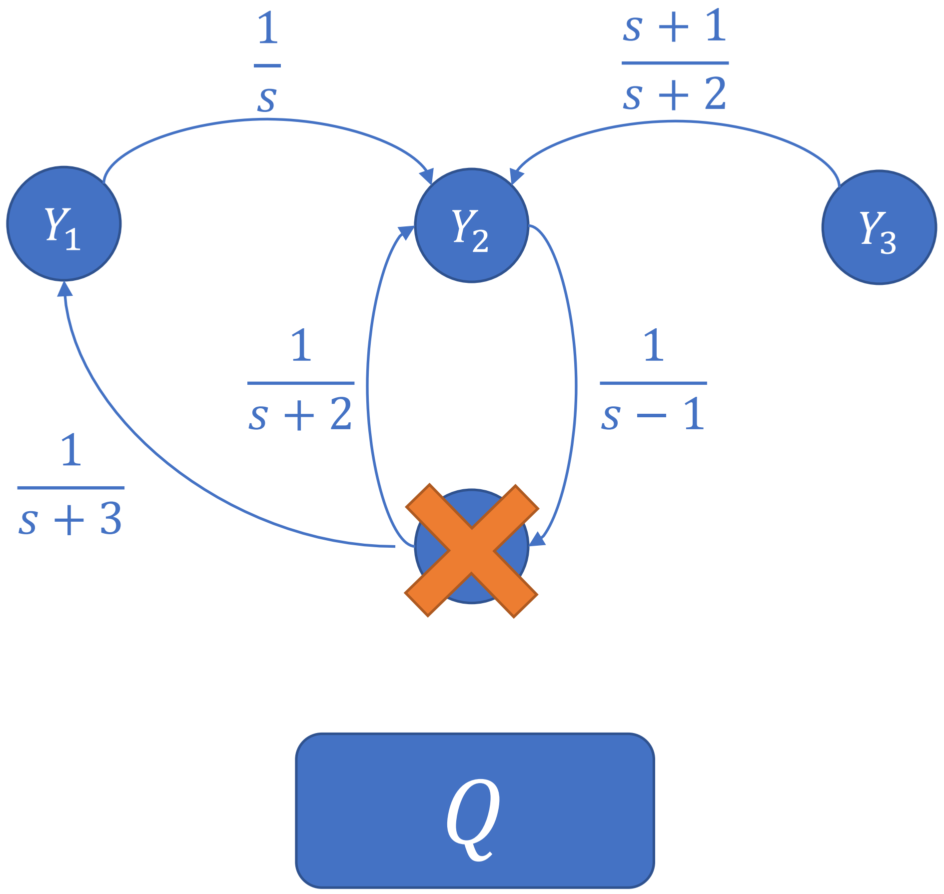

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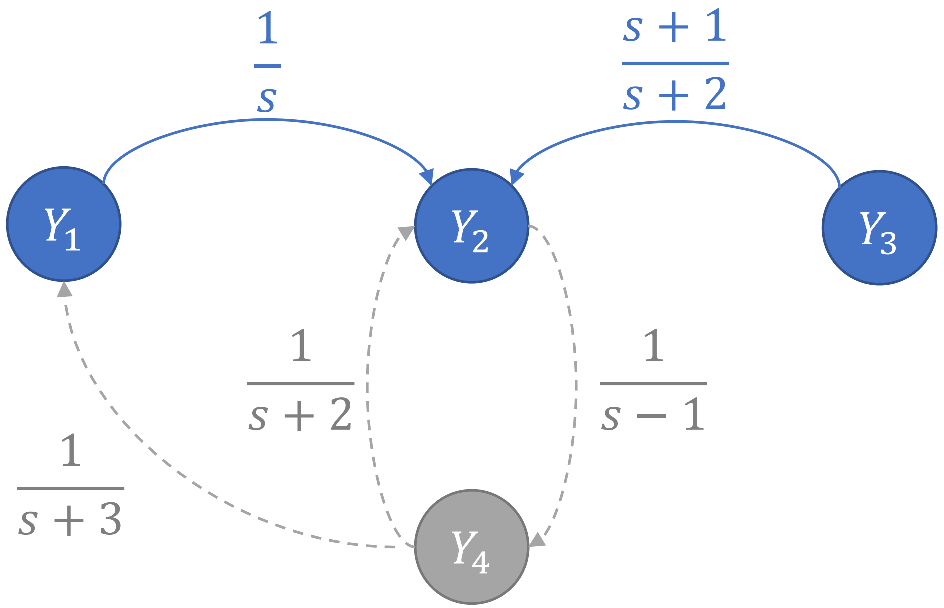

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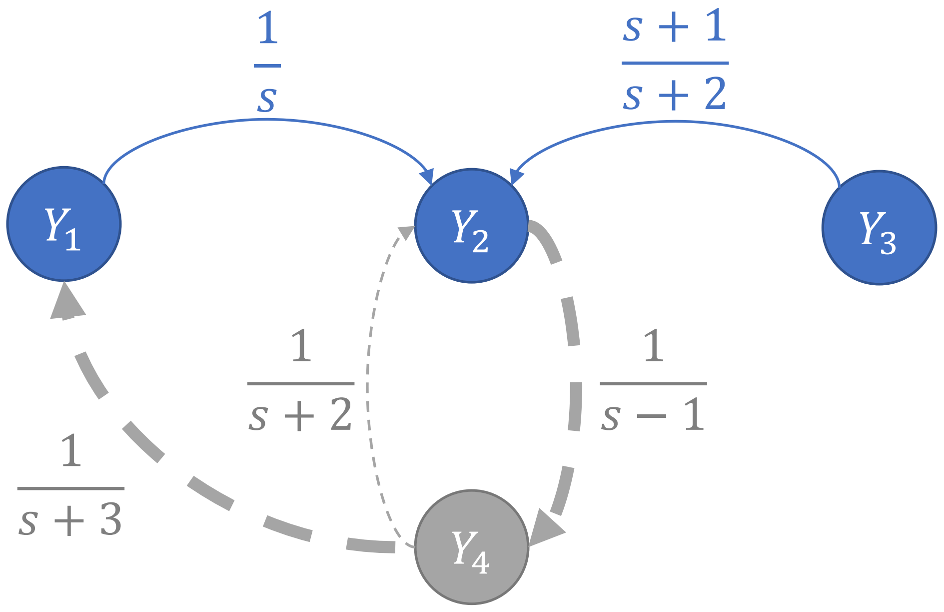

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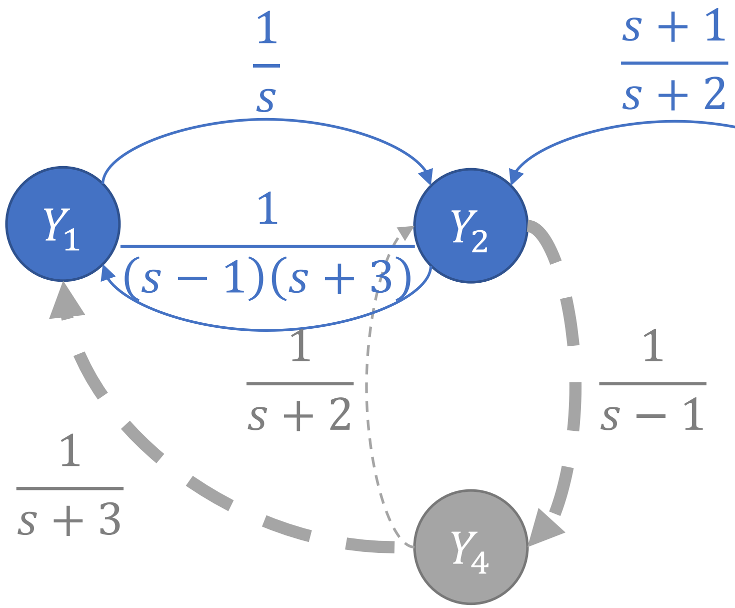

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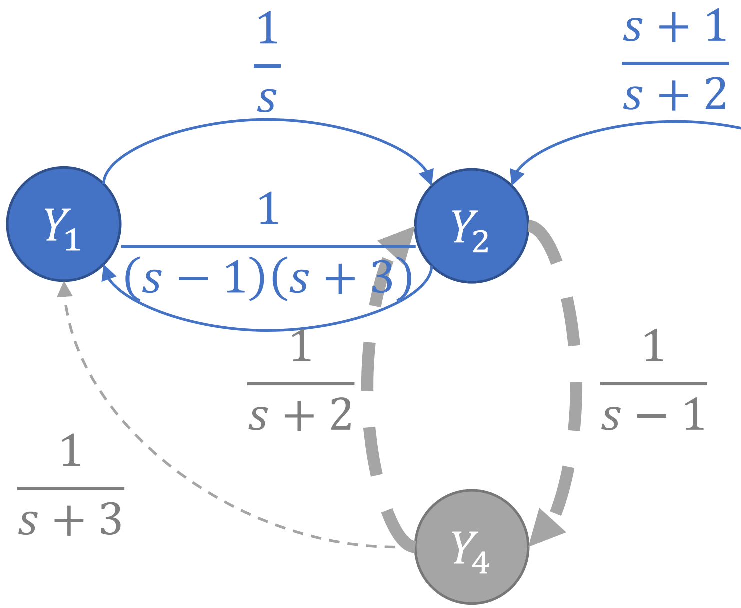

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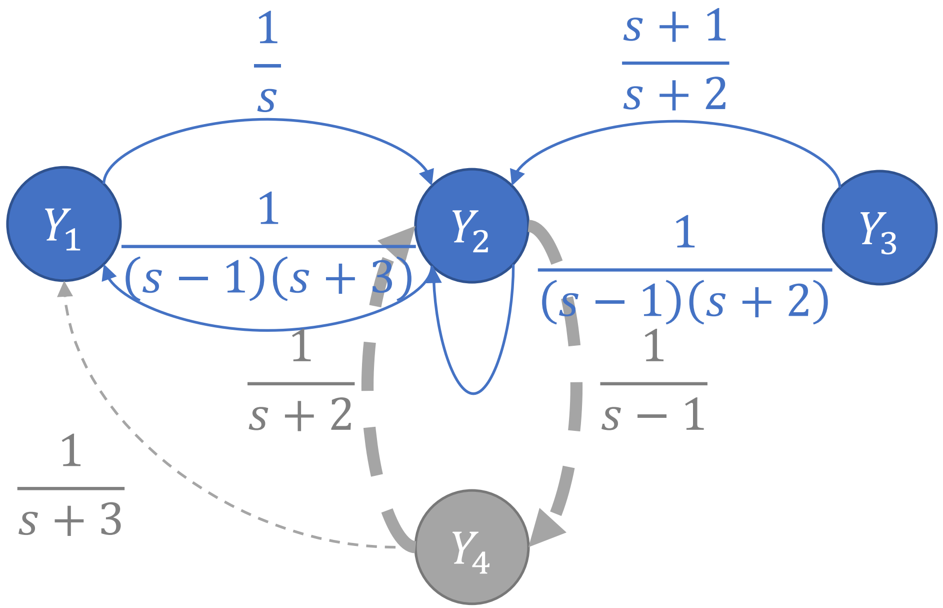

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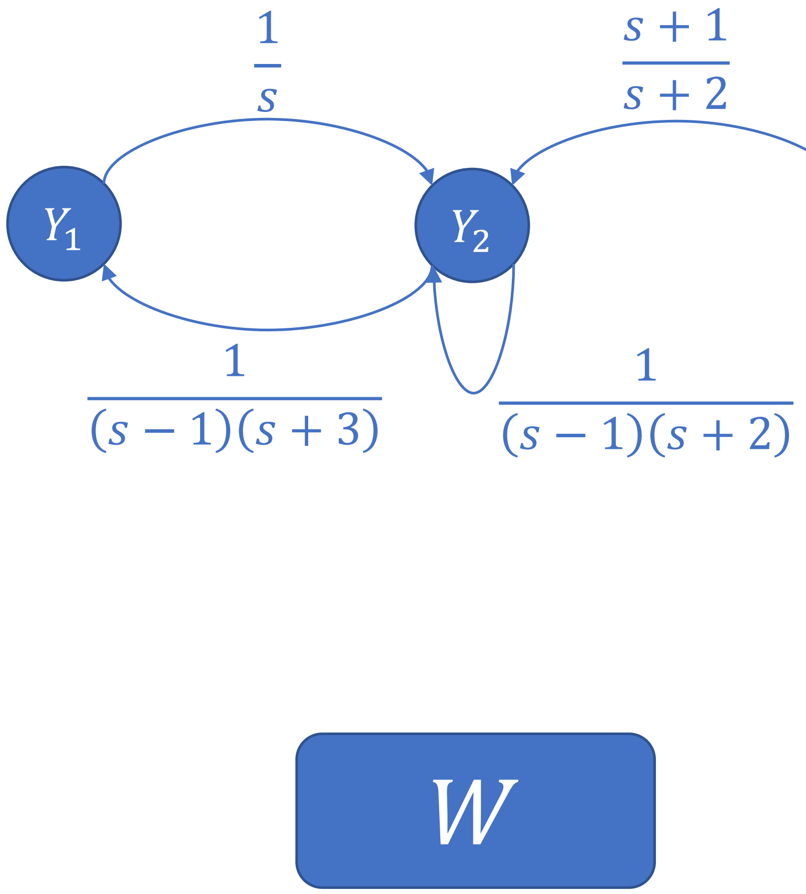

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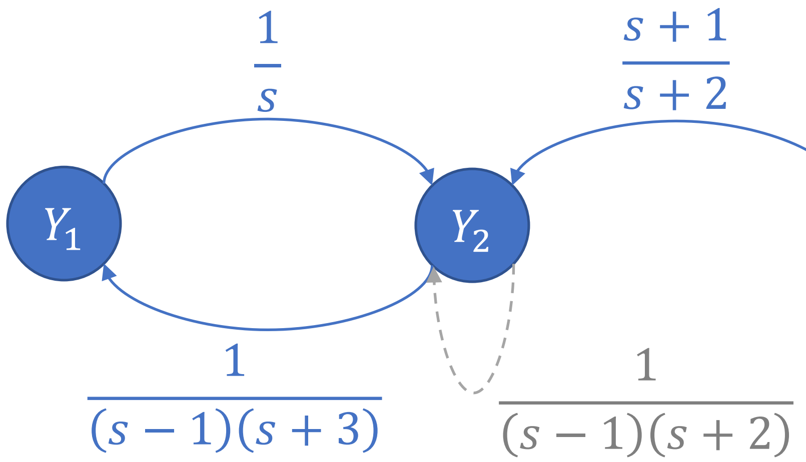

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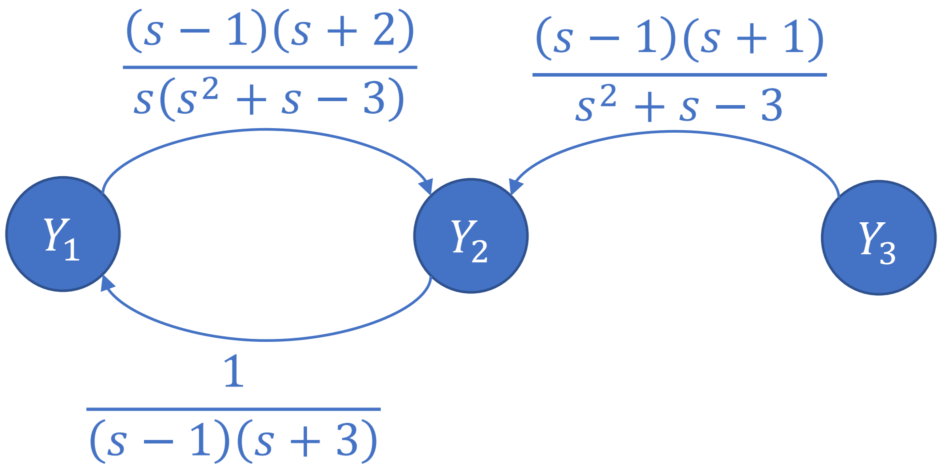
$W \rightarrow Q_S$

Abstraction: Example



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Q_S

Well-Posedness

Of DSFs and Abstractions

Understanding Well-Posedness

Jan Willems (1971)

- Well-posedness is essentially a modeling problem. It expresses that a mathematical model is, at least in principal, adequate as a description of a physical system.

Definition: Well-Posedness [1,2]

Let (Q, P) be a proper DSF defining the relationship $Y = QY + PU$. Then (Q, P) is said to be well-posed if:

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4. If conditions 1-3 hold for (Q, P), then **small perturbations to Q and P do not cause these conditions to fail**.

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Theorem (CDC 2017)

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- $\big(I - Q(\infty)\big) = \lim_{s\to\infty}\big(I - Q(s)\big)$ is non-singular

The Problem

- Let (Q_S, P_S) be the abstraction of (Q, P) with respect to S
- We wish to find the necessary *and* sufficient conditions for when (Q_S, P_S) is well-posed

Well-Posedness of Abstractions

Realization (Q, P)	Abstraction (Q_S, P_S)
• $\mathrm{I} - Q = \begin{bmatrix} I - Q_{SS} & -Q_{S\bar{S}} \\ -Q_{\bar{S}S} & I - Q_{\bar{S}\bar{S}} \end{bmatrix}$	• $I - W = (I - Q_{SS}) - Q_{S\bar{S}}(I - Q_{\bar{S}\bar{S}})^{-1}Q_{\bar{S}S}$ • $I - D_W = diag(I - W)$

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Lemma

- If $(I - Q)$, $(I - Q_{\bar{S}\bar{S}})$, and $(I - D_W)$ have proper inverses, then $(I - Q_S)$ has a proper inverse (the abstraction is well-posed)

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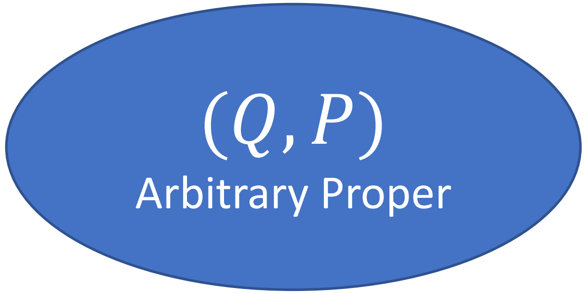
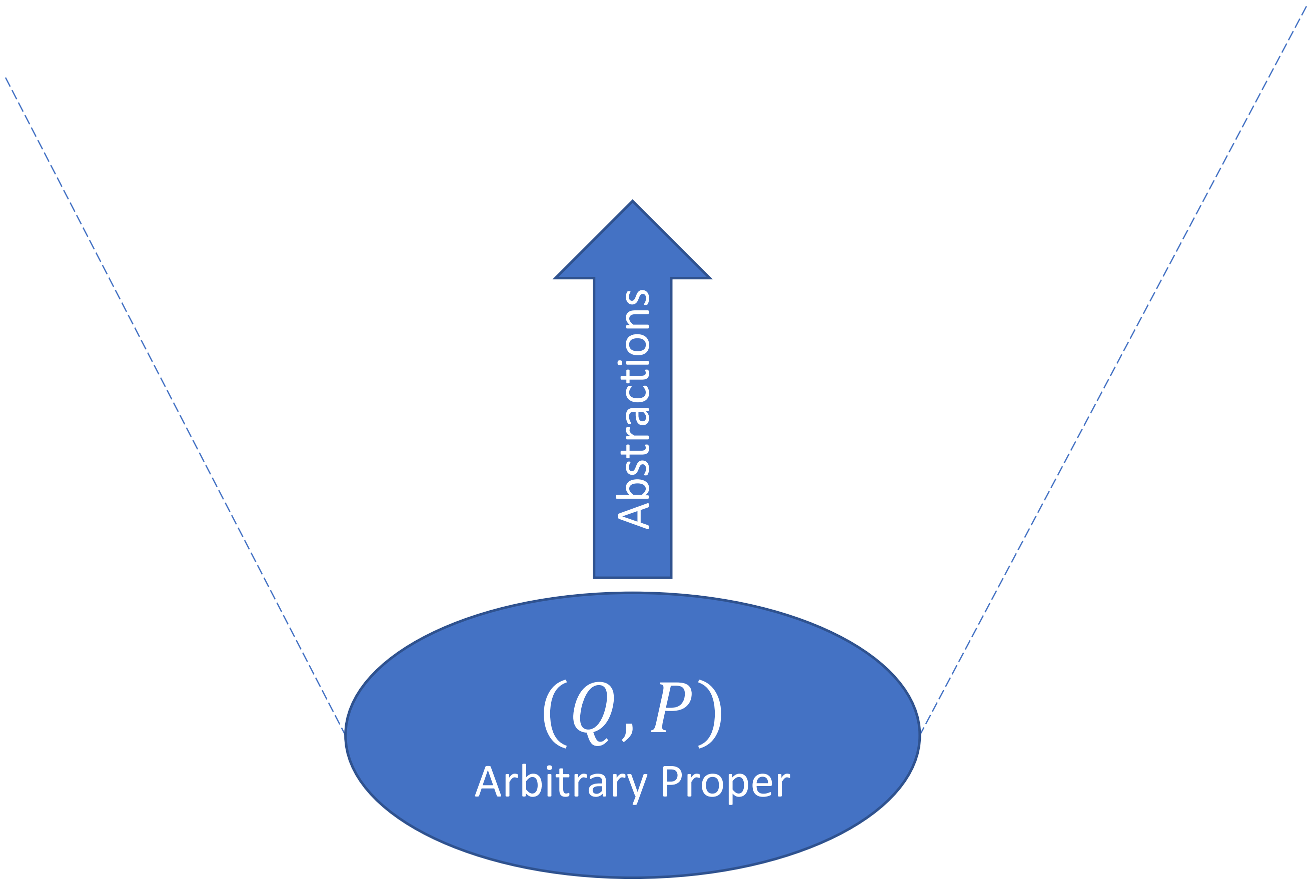
Theorem: Well-Posedness (Main Result)

- Let (Q_S, P_S) be a **representable** abstraction of (Q, P) with respect to S
- Then (Q_S, P_S) is well-posed if and only if (Q, P) is well-posed


(Q, P)
Arbitrary Proper



Abstractions
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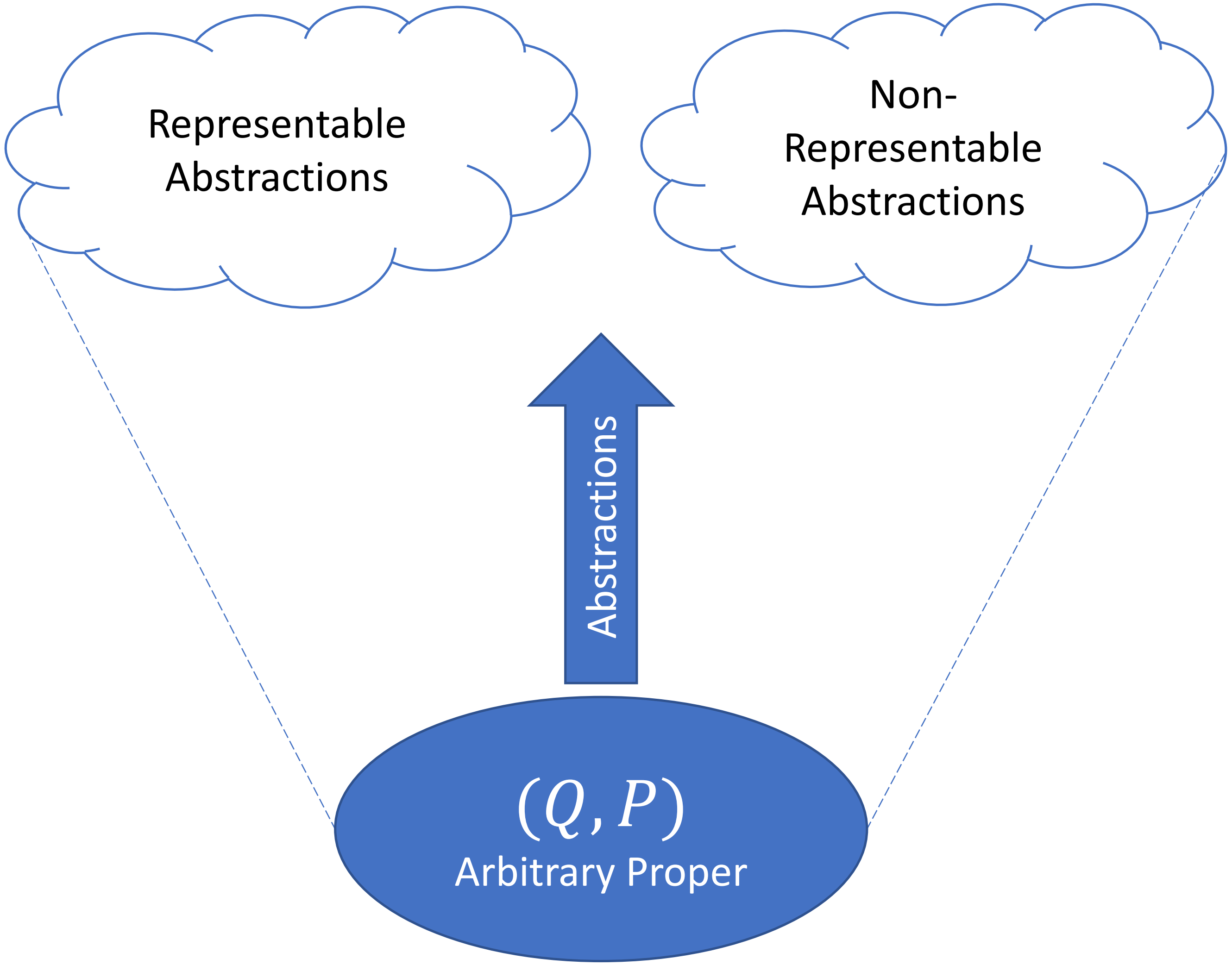
Representable Abstractions
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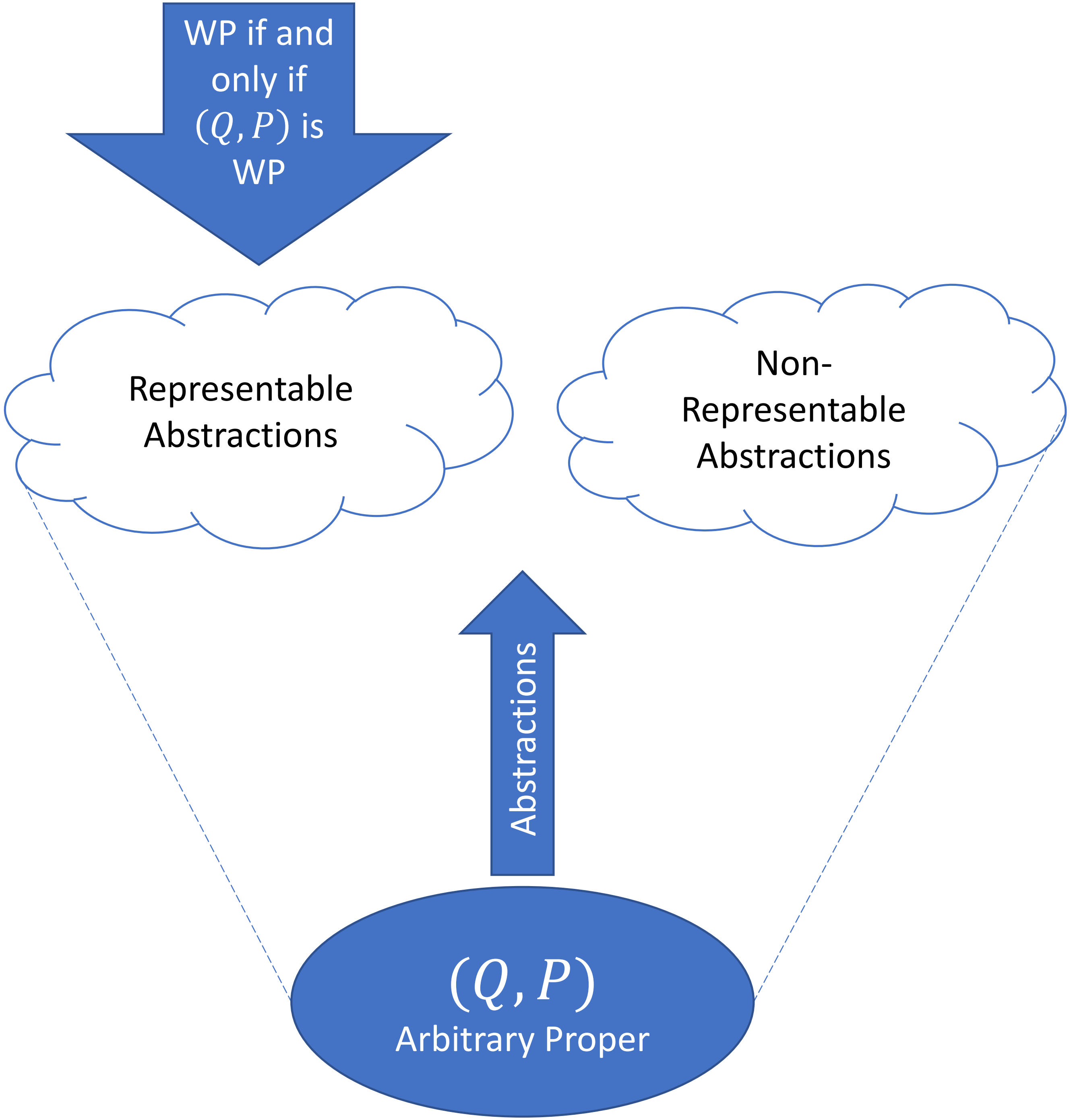
WP if and only if (Q, P) is WP
Representable Abstractions
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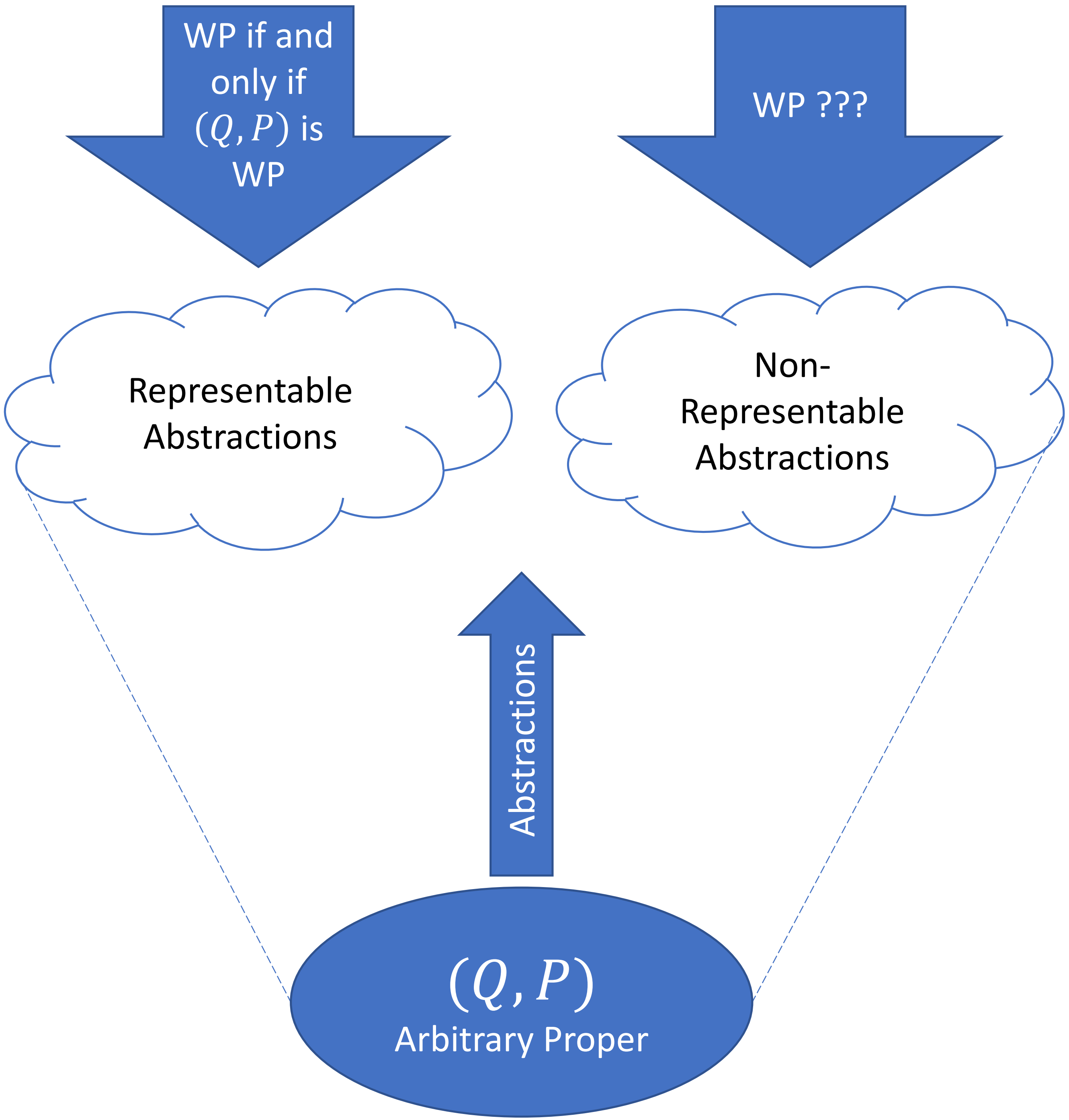
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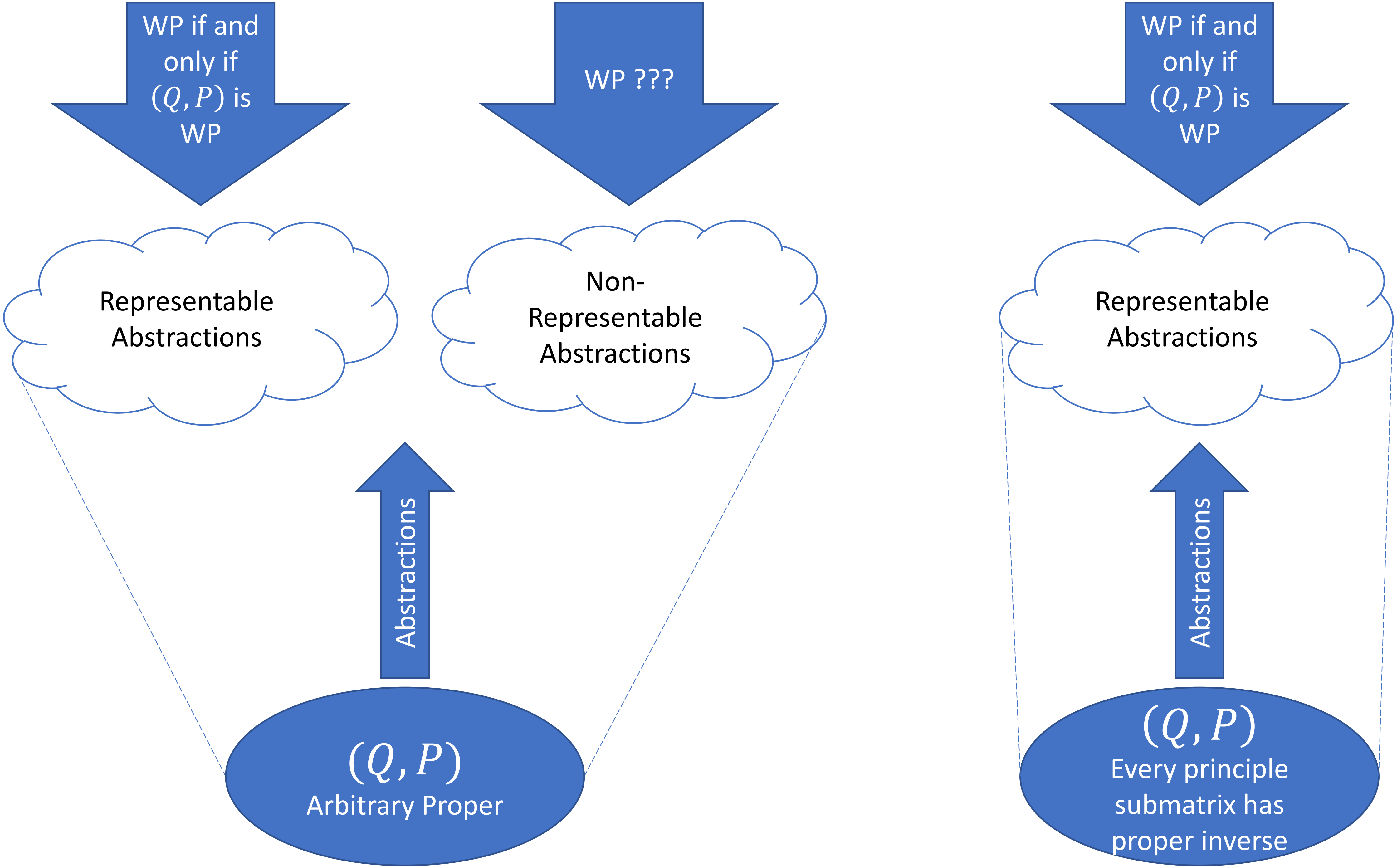
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(Q, P)
Every principle submatrix has proper inverse

Network Reconstruction

Motivation for Abstractions

Network Reconstruction Example – Insights

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    - Well-posedness results ensure reconstructed network is sensible
2. Representable abstractions of ill-posed networks are ill-posed

The Network Reconstruction Problem

- Suppose $Y = Q^*Y + P^*U$, for some fixed Q^* and P^*
- Given G, where $Y = GU$, with $G = [(I - Q^*)^{-1}P^*]$
- Find (Q^*, P^*)
- Need extra “a priori” information to solve

  - Full Conditions: J. Adebayo et. al. "Dynamical Structure Function Identifiability Conditions Enabling Signal Structure Reconstruction," Conference on Decision and Control, Maui, 2012.

  - Knowing P^* is diagonal is sufficient to find (Q^*, P^*) from G

Network Reconstruction Example

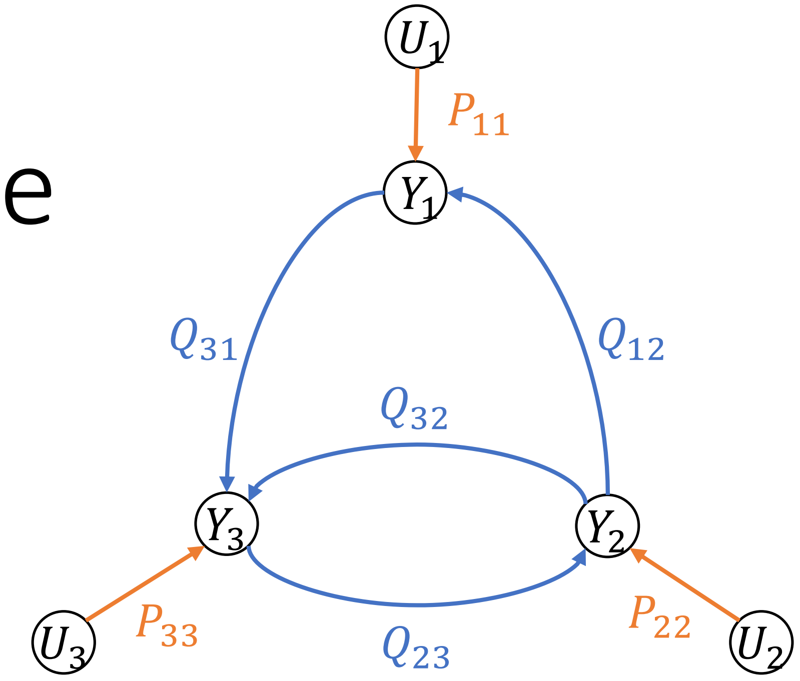
- Network reconstruction procedure from:

  J. Gonçalves and S. Warnick, "Necessary and Sufficient Conditions for Dynamical Structure Reconstruction of LTI Networks," IEEE Transactions on Automatic Control, Aug. 2008.

Network Reconstruction Example

- Suppose

$$Q = \begin{bmatrix} 0 & \frac{s+2}{s+1} & 0 \\ 0 & 0 & \frac{s+3}{s+4} \\ \frac{(s+1)(s+5)}{(s+2)(s+3)} & \frac{1}{s^2+2} & 0 \end{bmatrix}, \quad P = \begin{bmatrix} \frac{s+4}{s+1} & 0 & 0 \\ 0 & \frac{1}{s^2+2} & 0 \\ 0 & 0 & \frac{s+6}{s+3} \end{bmatrix}$$



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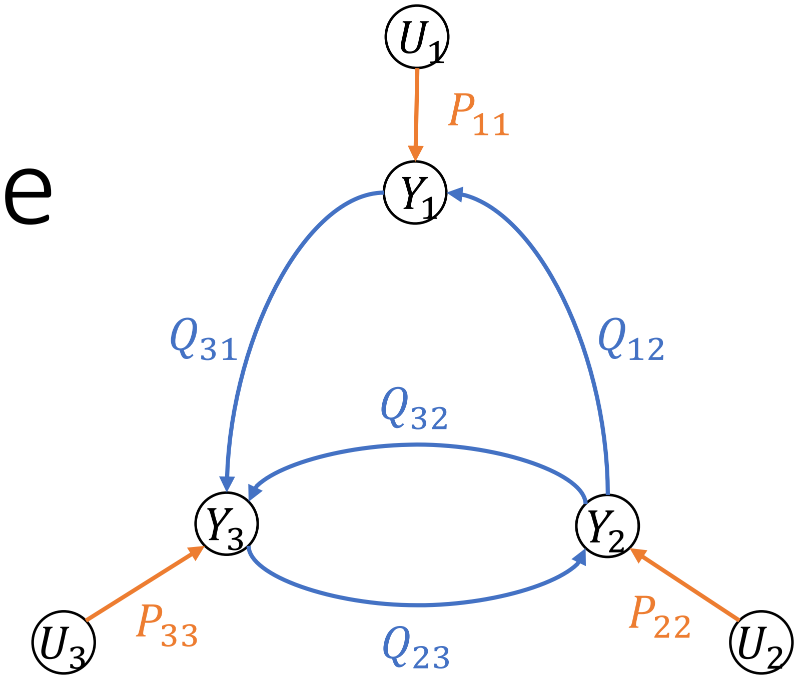

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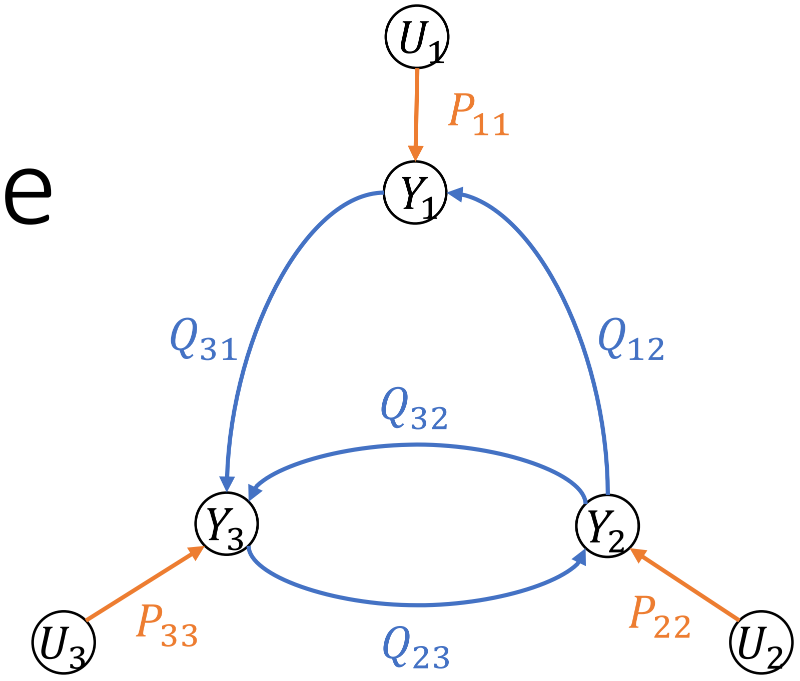
- We have:

$$G = (I - Q)^{-1}P = \begin{bmatrix} -\frac{(s+4)(s^3+4s^2+s+5)}{(s+1)(s^2+s+5)} & -\frac{(s+2)(s+4)}{(s+1)(s^2+s+5)} & -\frac{(s+2)(s+6)(s^2+2)}{(s+1)(s^2+s+5)} \\ -\frac{(s+4)(s+5)(s^2+2)}{(s+2)(s^2+s+5)} & -\frac{(s+4)}{(s^2+s+5)} & -\frac{(s+6)(s^2+2)}{(s^2+s+5)} \\ -\frac{(s+4)^2(s+5)(s^2+2)}{(s+2)(s+3)(s^2+s+5)} & -\frac{(s+4)(s^3+5s^2+3s+13)}{(s+3)(s^2+2)(s^2+s+5)} & -\frac{(s+4)(s+6)(s^2+2)}{(s+3)(s^2+s+5)} \end{bmatrix}$$

Network Reconstruction Example



- Unknown

$$Q = \begin{bmatrix} 0 & Q_{12} & Q_{13} \\ Q_{21} & 0 & Q_{23} \\ Q_{31} & Q_{32} & 0 \end{bmatrix}, \qquad P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}$$

- We have:

$$G = (I - Q)^{-1} P = \begin{bmatrix} -\frac{(s+4)(s^3+4s^2+s+5)}{(s+1)(s^2+s+5)} & -\frac{(s+2)(s+4)}{(s+1)(s^2+s+5)} & -\frac{(s+2)(s+6)(s^2+2)}{(s+1)(s^2+s+5)} \\ -\frac{(s+4)(s+5)(s^2+2)}{(s+2)(s^2+s+5)} & -\frac{(s+4)}{(s^2+s+5)} & -\frac{(s+6)(s^2+2)}{(s^2+s+5)} \\ -\frac{(s+4)^2(s+5)(s^2+2)}{(s+2)(s+3)(s^2+s+5)} & -\frac{(s+4)(s^3+5s^2+3s+13)}{(s+3)(s^2+2)(s^2+s+5)} & -\frac{(s+4)(s+6)(s^2+2)}{(s+3)(s^2+s+5)} \end{bmatrix}$$

Network Reconstruction General Solution

- Solution (where Q is $p \times p$ and P, G are $p \times m$):

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$$G = QG + P$$

Network Reconstruction General Solution

- Solution (where Q is $p \times p$ and P, G are $p \times m$):

$$
\begin{aligned}
G &= \left(I_p - Q\right)^{-1} P \qquad \Longrightarrow \\
G &= QG + P \\
&= [Q \quad P] \begin{bmatrix} G \\ I_p \end{bmatrix}
\end{aligned}
$$

Network Reconstruction General Solution

- Solution (where Q is $p \times p$ and P, G are $p \times m$):

$$G = \left(I_p - Q\right)^{-1} P \quad \Longrightarrow$$

$$G = QG + P$$

$$= \begin{bmatrix} Q & P \end{bmatrix} \begin{bmatrix} G \\ I_p \end{bmatrix} \quad \Longrightarrow$$

$$G' = \begin{bmatrix} G' & I_p \end{bmatrix} \begin{bmatrix} Q' \\ P' \end{bmatrix}$$

Network Reconstruction General Solution

- Solution (where Q is $p \times p$ and P, G are $p \times m$):

$$G = \left(I_p - Q\right)^{-1} P \qquad \Longrightarrow$$

$$G = QG + P$$

$$= [Q \quad P] \begin{bmatrix} G \\ I_p \end{bmatrix} \quad \Longrightarrow$$

$$G' = [G' \quad I_p] \begin{bmatrix} Q' \\ P' \end{bmatrix} \quad \Longrightarrow$$

$$\vec{g} = L\vec{\theta}$$

Network Reconstruction Example

$$
\underbrace{\begin{bmatrix} G_{11} \\ G_{21} \\ G_{31} \\ G_{12} \\ G_{22} \\ G_{32} \\ G_{13} \\ G_{23} \\ G_{33} \end{bmatrix}}_{\vec{g}}
=
\underbrace{\begin{bmatrix}
G_{11} & G_{21} & G_{31} & 1 & 0 & 0 & & & & & & & & & & & & \\
& & & & & & G_{11} & G_{21} & G_{31} & 1 & 0 & 0 & & & & & & \\
& & & & & & & & & & & & G_{11} & G_{21} & G_{31} & 1 & 0 & 0 \\
G_{12} & G_{22} & G_{32} & 0 & 1 & 0 & & & & & & & & & & & & \\
& & & & & & G_{12} & G_{22} & G_{32} & 0 & 1 & 0 & & & & & & \\
& & & & & & & & & & & & G_{12} & G_{22} & G_{32} & 0 & 1 & 0 \\
G_{13} & G_{23} & G_{33} & 0 & 0 & 1 & & & & & & & & & & & & \\
& & & & & & G_{13} & G_{23} & G_{33} & 0 & 0 & 1 & & & & & & \\
& & & & & & & & & & & & G_{13} & G_{23} & G_{33} & 0 & 0 & 1
\end{bmatrix}}_{L}
\underbrace{\begin{bmatrix} 0 \\ Q_{21} \\ Q_{31} \\ P_{11} \\ P_{12} \\ P_{13} \\ Q_{12} \\ 0 \\ Q_{32} \\ P_{21} \\ P_{22} \\ P_{33} \\ Q_{13} \\ Q_{23} \\ 0 \\ P_{31} \\ P_{32} \\ P_{33} \end{bmatrix}}_{\vec{\theta}}
$$

$$
G = \begin{bmatrix}
-\frac{(s+4)(s^3+4s^2+s+5)}{(s+1)(s^2+s+5)} & -\frac{(s+2)(s+4)}{(s+1)(s^2+s+5)} & -\frac{(s+2)(s+6)(s^2+2)}{(s+1)(s^2+s+5)} \\
-\frac{(s+4)(s+5)(s^2+2)}{(s+2)(s^2+s+5)} & -\frac{(s+4)}{(s^2+s+5)} & -\frac{(s+6)(s^2+2)}{(s^2+s+5)} \\
-\frac{(s+4)^2(s+5)(s^2+2)}{(s+2)(s+3)(s^2+s+5)} & -\frac{(s+4)(s^3+5s^2+3s+13)}{(s+3)(s^2+2)(s^2+s+5)} & -\frac{(s+4)(s+6)(s^2+2)}{(s+3)(s^2+s+5)}
\end{bmatrix}
$$

Network Reconstruction Example

$pm = 9$
Equations

$$\underbrace{\begin{bmatrix} G_{11} \\ G_{21} \\ G_{31} \\ G_{12} \\ G_{22} \\ G_{32} \\ G_{13} \\ G_{23} \\ G_{33} \end{bmatrix}}_{\vec{g}} = \underbrace{\begin{bmatrix} G_{11} & G_{21} & G_{31} & 1 & 0 & 0 & & & & & & & & & & & & \\ & & & & & & G_{11} & G_{21} & G_{31} & 1 & 0 & 0 & & & & & & \\ & & & & & & & & & & & & G_{11} & G_{21} & G_{31} & 1 & 0 & 0 \\ G_{12} & G_{22} & G_{32} & 0 & 1 & 0 & & & & & & & & & & & & \\ & & & & & & G_{12} & G_{22} & G_{32} & 0 & 1 & 0 & & & & & & \\ & & & & & & & & & & & & G_{12} & G_{22} & G_{32} & 0 & 1 & 0 \\ G_{13} & G_{23} & G_{33} & 0 & 0 & 1 & & & & & & & & & & & & \\ & & & & & & G_{13} & G_{23} & G_{33} & 0 & 0 & 1 & & & & & & \\ & & & & & & & & & & & & G_{13} & G_{23} & G_{33} & 0 & 0 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 0 \\ Q_{21} \\ Q_{31} \\ P_{11} \\ P_{12} \\ P_{13} \\ Q_{12} \\ 0 \\ Q_{32} \\ P_{21} \\ P_{22} \\ P_{33} \\ Q_{13} \\ Q_{23} \\ 0 \\ P_{31} \\ P_{32} \\ P_{33} \end{bmatrix}}_{\vec{\theta}}$$

$$G = \begin{bmatrix} -\frac{(s+4)(s^3+4s^2+s+5)}{(s+1)(s^2+s+5)} & -\frac{(s+2)(s+4)}{(s+1)(s^2+s+5)} & -\frac{(s+2)(s+6)(s^2+2)}{(s+1)(s^2+s+5)} \\ -\frac{(s+4)(s+5)(s^2+2)}{(s+2)(s^2+s+5)} & -\frac{(s+4)}{(s^2+s+5)} & -\frac{(s+6)(s^2+2)}{(s^2+s+5)} \\ -\frac{(s+4)^2(s+5)(s^2+2)}{(s+2)(s+3)(s^2+s+5)} & -\frac{(s+4)(s^3+5s^2+3s+13)}{(s+3)(s^2+2)(s^2+s+5)} & -\frac{(s+4)(s+6)(s^2+2)}{(s+3)(s^2+s+5)} \end{bmatrix}$$

Network Reconstruction Example

$pm = 9$
Equations

$$\underbrace{\begin{bmatrix} G_{11} \\ G_{21} \\ G_{31} \\ G_{12} \\ G_{22} \\ G_{32} \\ G_{13} \\ G_{23} \\ G_{33} \end{bmatrix}}_{\vec{g}} = \underbrace{\begin{bmatrix}
G_{11} & G_{21} & G_{31} & 1 & 0 & 0 & & & & & & & & & & & & \\
& & & & & & G_{11} & G_{21} & G_{31} & 1 & 0 & 0 & & & & & & \\
& & & & & & & & & & & & G_{11} & G_{21} & G_{31} & 1 & 0 & 0 \\
G_{12} & G_{22} & G_{32} & 0 & 1 & 0 & & & & & & & & & & & & \\
& & & & & & G_{12} & G_{22} & G_{32} & 0 & 1 & 0 & & & & & & \\
& & & & & & & & & & & & G_{12} & G_{22} & G_{32} & 0 & 1 & 0 \\
G_{13} & G_{23} & G_{33} & 0 & 0 & 1 & & & & & & & & & & & & \\
& & & & & & G_{13} & G_{23} & G_{33} & 0 & 0 & 1 & & & & & & \\
& & & & & & & & & & & & G_{13} & G_{23} & G_{33} & 0 & 0 & 1
\end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 0 \\ Q_{21} \\ Q_{31} \\ P_{11} \\ P_{12} \\ P_{13} \\ Q_{12} \\ 0 \\ Q_{32} \\ P_{21} \\ P_{22} \\ P_{33} \\ Q_{13} \\ Q_{23} \\ 0 \\ P_{31} \\ P_{32} \\ P_{33} \end{bmatrix}}_{\vec{\theta}}$$

$p^2 + pm = 18$
Unknowns

$$G = \begin{bmatrix}
-\frac{(s+4)(s^3+4s^2+s+5)}{(s+1)(s^2+s+5)} & -\frac{(s+2)(s+4)}{(s+1)(s^2+s+5)} & -\frac{(s+2)(s+6)(s^2+2)}{(s+1)(s^2+s+5)} \\
-\frac{(s+4)(s+5)(s^2+2)}{(s+2)(s^2+s+5)} & -\frac{(s+4)}{(s^2+s+5)} & -\frac{(s+6)(s^2+2)}{(s^2+s+5)} \\
-\frac{(s+4)^2(s+5)(s^2+2)}{(s+2)(s+3)(s^2+s+5)} & -\frac{(s+4)(s^3+5s^2+3s+13)}{(s+3)(s^2+2)(s^2+s+5)} & -\frac{(s+4)(s+6)(s^2+2)}{(s+3)(s^2+s+5)}
\end{bmatrix}$$

Network Reconstruction Example

$$\underbrace{\begin{bmatrix} G_{11} \\ G_{21} \\ G_{31} \\ G_{12} \\ G_{22} \\ G_{32} \\ G_{13} \\ G_{23} \\ G_{33} \end{bmatrix}}_{\vec{g}} = \underbrace{\left[\begin{array}{cccccccccccccccccc} G_{11} & G_{21} & G_{31} & 1 & 0 & 0 & & & & & & & & & & & & \\ G_{12} & G_{22} & G_{32} & 0 & 1 & 0 & & & & & & & & & & & & \\ G_{13} & G_{23} & G_{33} & 0 & 0 & 1 & & & & & & & & & & & & \\ & & & & & & G_{11} & G_{21} & G_{31} & 1 & 0 & 0 & & & & & & \\ & & & & & & G_{12} & G_{22} & G_{32} & 0 & 1 & 0 & & & & & & \\ & & & & & & G_{13} & G_{23} & G_{33} & 0 & 0 & 1 & & & & & & \\ & & & & & & & & & & & & G_{11} & G_{21} & G_{31} & 1 & 0 & 0 \\ & & & & & & & & & & & & G_{12} & G_{22} & G_{32} & 0 & 1 & 0 \\ & & & & & & & & & & & & G_{13} & G_{23} & G_{33} & 0 & 0 & 1 \end{array}\right]}_{L} \underbrace{\begin{bmatrix} 0 \\ Q_{21} \\ Q_{31} \\ P_{11} \\ P_{12} \\ P_{13} \\ Q_{12} \\ 0 \\ Q_{32} \\ P_{21} \\ P_{22} \\ P_{33} \\ Q_{13} \\ Q_{23} \\ 0 \\ P_{31} \\ P_{32} \\ P_{33} \end{bmatrix}}_{\vec{\theta}}$$

$pm = 9$
Equations

$p^2 - p + pm = 15$
Unknowns

$$G = \begin{bmatrix} -\frac{(s+4)(s^3+4s^2+s+5)}{(s+1)(s^2+s+5)} & -\frac{(s+2)(s+4)}{(s+1)(s^2+s+5)} & -\frac{(s+2)(s+6)(s^2+2)}{(s+1)(s^2+s+5)} \\ -\frac{(s+4)(s+5)(s^2+2)}{(s+2)(s^2+s+5)} & -\frac{(s+4)}{(s^2+s+5)} & -\frac{(s+6)(s^2+2)}{(s^2+s+5)} \\ -\frac{(s+4)^2(s+5)(s^2+2)}{(s+2)(s+3)(s^2+s+5)} & -\frac{(s+4)(s^3+5s^2+3s+13)}{(s+3)(s^2+2)(s^2+s+5)} & -\frac{(s+4)(s+6)(s^2+2)}{(s+3)(s^2+s+5)} \end{bmatrix}$$

Network Reconstruction Example

$pm = 9$
Equations

$$\underbrace{\begin{bmatrix} G_{11} \\ G_{21} \\ G_{31} \\ G_{12} \\ G_{22} \\ G_{32} \\ G_{13} \\ G_{23} \\ G_{33} \end{bmatrix}}_{\vec{g}} = \underbrace{\begin{bmatrix} G_{11} & G_{21} & G_{31} & 1 & 0 & 0 & & & & & & & & & & & & \\ G_{12} & G_{22} & G_{32} & 0 & 1 & 0 & & & & & & & & & & & & \\ G_{13} & G_{23} & G_{33} & 0 & 0 & 1 & & & & & & & & & & & & \\ & & & & & & G_{11} & G_{21} & G_{31} & 1 & 0 & 0 & & & & & & \\ & & & & & & G_{12} & G_{22} & G_{32} & 0 & 1 & 0 & & & & & & \\ & & & & & & G_{13} & G_{23} & G_{33} & 0 & 0 & 1 & & & & & & \\ & & & & & & & & & & & & G_{11} & G_{21} & G_{31} & 1 & 0 & 0 \\ & & & & & & & & & & & & G_{12} & G_{22} & G_{32} & 0 & 1 & 0 \\ & & & & & & & & & & & & G_{13} & G_{23} & G_{33} & 0 & 0 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 0 \\ Q_{21} \\ Q_{31} \\ P_{11} \\ P_{12} \\ P_{13} \\ Q_{12} \\ 0 \\ Q_{32} \\ P_{21} \\ P_{22} \\ P_{33} \\ Q_{13} \\ Q_{23} \\ 0 \\ P_{31} \\ P_{32} \\ P_{33} \end{bmatrix}}_{\vec{\theta}}$$

$p^2 - p + pm = 15$
Unknowns

$$G = \begin{bmatrix} -\frac{(s+4)(s^3+4s^2+s+5)}{(s+1)(s^2+s+5)} & -\frac{(s+2)(s+4)}{(s+1)(s^2+s+5)} & -\frac{(s+2)(s+6)(s^2+2)}{(s+1)(s^2+s+5)} \\ -\frac{(s+4)(s+5)(s^2+2)}{(s+2)(s^2+s+5)} & -\frac{(s+4)}{(s^2+s+5)} & -\frac{(s+6)(s^2+2)}{(s^2+s+5)} \\ -\frac{(s+4)^2(s+5)(s^2+2)}{(s+2)(s+3)(s^2+s+5)} & -\frac{(s+4)(s^3+5s^2+3s+13)}{(s+3)(s^2+2)(s^2+s+5)} & -\frac{(s+4)(s+6)(s^2+2)}{(s+3)(s^2+s+5)} \end{bmatrix}$$

Suppose we knew that P is diagonal ($P_{ij} = 0$ for $i \neq j$)

Network Reconstruction Example

$pm = 9$
Equations

$$\underbrace{\begin{bmatrix} G_{11} \\ G_{21} \\ G_{31} \\ G_{12} \\ G_{22} \\ G_{32} \\ G_{13} \\ G_{23} \\ G_{33} \end{bmatrix}}_{\vec{g}} = \underbrace{\begin{bmatrix} G_{11} & G_{21} & G_{31} & 1 & 0 & 0 & & & & & & & & & & & & \\ G_{12} & G_{22} & G_{32} & 0 & 1 & 0 & & & & & & & & & & & & \\ G_{13} & G_{23} & G_{33} & 0 & 0 & 1 & & & & & & & & & & & & \\ & & & & & & G_{11} & G_{21} & G_{31} & 1 & 0 & 0 & & & & & & \\ & & & & & & G_{12} & G_{22} & G_{32} & 0 & 1 & 0 & & & & & & \\ & & & & & & G_{13} & G_{23} & G_{33} & 0 & 0 & 1 & & & & & & \\ & & & & & & & & & & & & G_{11} & G_{21} & G_{31} & 1 & 0 & 0 \\ & & & & & & & & & & & & G_{12} & G_{22} & G_{32} & 0 & 1 & 0 \\ & & & & & & & & & & & & G_{13} & G_{23} & G_{33} & 0 & 0 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 0 \\ Q_{21} \\ Q_{31} \\ P_{11} \\ P_{12} \\ P_{13} \\ Q_{12} \\ 0 \\ Q_{32} \\ P_{21} \\ P_{22} \\ P_{33} \\ Q_{13} \\ Q_{23} \\ 0 \\ P_{31} \\ P_{32} \\ P_{33} \end{bmatrix}}_{\vec{\theta}}$$

$p^2 - p + pm - (p^2 - p)$
$= pm = 9$ Unknowns

$$G = \begin{bmatrix} -\frac{(s+4)(s^3+4s^2+s+5)}{(s+1)(s^2+s+5)} & -\frac{(s+2)(s+4)}{(s+1)(s^2+s+5)} & -\frac{(s+2)(s+6)(s^2+2)}{(s+1)(s^2+s+5)} \\ -\frac{(s+4)(s+5)(s^2+2)}{(s+2)(s^2+s+5)} & -\frac{(s+4)}{(s^2+s+5)} & -\frac{(s+6)(s^2+2)}{(s^2+s+5)} \\ -\frac{(s+4)^2(s+5)(s^2+2)}{(s+2)(s+3)(s^2+s+5)} & -\frac{(s+4)(s^3+5s^2+3s+13)}{(s+3)(s^2+2)(s^2+s+5)} & -\frac{(s+4)(s+6)(s^2+2)}{(s+3)(s^2+s+5)} \end{bmatrix}$$

Suppose we knew that P is diagonal ($P_{ij} = 0$ for $i \neq j$)
There are $p^2 - p$ off-diagonal entries

Network Reconstruction Example

- We now have $\vec{g} = L\vec{\theta}$, with L square

Network Reconstruction Example

- We now have $\vec{g} = L\vec{\theta}$, with L square
- Find $\vec{\theta} = L^{-1}\vec{g}$, giving:

$$\begin{bmatrix} Q_{21} \\ Q_{31} \\ P_{11} \\ Q_{12} \\ Q_{32} \\ P_{22} \\ Q_{13} \\ Q_{23} \\ P_{33} \end{bmatrix} = \begin{bmatrix} \frac{s+2}{s+1} \\ 0 \\ \frac{s+4}{s+1} \\ 0 \\ \frac{s+3}{s+4} \\ \frac{1}{s^2+2} \\ \frac{(s+1)(s+5)}{(s+2)(s+3)} \\ \frac{1}{s^2+2} \\ \frac{s+6}{s+3} \end{bmatrix}$$

Network Reconstruction Example

- We now have $\vec{g} = L\vec{\theta}$, with L square
- Find $\vec{\theta} = L^{-1}\vec{g}$, giving:

$$\begin{bmatrix} Q_{21} \\ Q_{31} \\ P_{11} \\ Q_{12} \\ Q_{32} \\ P_{22} \\ Q_{13} \\ Q_{23} \\ P_{33} \end{bmatrix} = \begin{bmatrix} \frac{s+2}{s+1} \\ 0 \\ \frac{s+4}{s+1} \\ 0 \\ \frac{s+3}{s+4} \\ \frac{1}{s^2+2} \\ \frac{(s+1)(s+5)}{(s+2)(s+3)} \\ \frac{1}{s^2+2} \\ \frac{s+6}{s+3} \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & \frac{s+2}{s+1} & 0 \\ 0 & 0 & \frac{s+3}{s+4} \\ \frac{(s+1)(s+5)}{(s+2)(s+3)} & \frac{1}{s^2+2} & 0 \end{bmatrix}, \qquad P = \begin{bmatrix} \frac{s+4}{s+1} & 0 & 0 \\ 0 & \frac{1}{s^2+2} & 0 \\ 0 & 0 & \frac{s+6}{s+3} \end{bmatrix}$$

Abstractions and Network Reconstruction

- Recall that an abstraction is a new network with $|S| < p$ outputs
  - Cost to identify base network: $p^2 - p$
  - Cost to identify abstraction: $|S|^2 - |S| < p^2 - p$

Abstraction and Reconstruction Example

- Base Network

$$Q=\begin{bmatrix} 0 & \frac{s+2}{s+1} & 0 \\ 0 & 0 & \frac{s+3}{s+4} \\ \frac{(s+1)(s+5)}{(s+2)(s+3)} & \frac{1}{s^2+2} & 0 \end{bmatrix}, \qquad P=\begin{bmatrix} \frac{s+4}{s+1} & 0 & 0 \\ 0 & \frac{1}{s^2+2} & 0 \\ 0 & 0 & \frac{s+6}{s+3} \end{bmatrix}$$

$$G=\begin{bmatrix} -\frac{(s+4)(s^3+4s^2+s+5)}{(s+1)(s^2+s+5)} & -\frac{(s+2)(s+4)}{(s+1)(s^2+s+5)} & -\frac{(s+2)(s+6)(s^2+2)}{(s+1)(s^2+s+5)} \\ -\frac{(s+4)(s+5)(s^2+2)}{(s+2)(s^2+s+5)} & -\frac{(s+4)}{(s^2+s+5)} & -\frac{(s+6)(s^2+2)}{(s^2+s+5)} \\ -\frac{(s+4)^2(s+5)(s^2+2)}{(s+2)(s+3)(s^2+s+5)} & -\frac{(s+4)(s^3+5s^2+3s+13)}{(s+3)(s^2+2)(s^2+s+5)} & -\frac{(s+4)(s+6)(s^2+2)}{(s+3)(s^2+s+5)} \end{bmatrix}$$

- Abstraction with $S=\{1,2\}$

$$Q_S=\begin{bmatrix} 0 & \frac{(s+2)}{(s+1)} \\ \frac{(s+1)(s+5)(s^2+2)}{(s+2)(s^3+4s^2+s+5)} & 0 \end{bmatrix},$$

$$P_S=\begin{bmatrix} \frac{(s+4)}{(s+1)} & 0 & 0 \\ 0 & \frac{(s+4)}{(s^3+4s^2+s+5)} & \frac{(s+6)(s^2+2)}{(s^3+4s^2+s+5)} \end{bmatrix}$$

$$G_S=\begin{bmatrix} -\frac{(s+4)(s^3+4s^2+s+5)}{(s+1)(s^2+s+5)} & -\frac{(s+2)(s+4)}{(s+1)(s^2+s+5)} & -\frac{(s+2)(s+6)(s^2+2)}{(s+1)(s^2+s+5)} \\ -\frac{(s+4)(s+5)(s^2+2)}{(s+2)(s^2+s+5)} & -\frac{(s+4)}{(s^2+s+5)} & -\frac{(s+6)(s^2+2)}{(s^2+s+5)} \end{bmatrix}$$

Abstraction and Reconstruction Example

- Base Network

$$Q = \begin{bmatrix} 0 & \frac{s+2}{s+1} & 0 \\ 0 & 0 & \frac{s+3}{s+4} \\ \frac{(s+1)(s+5)}{(s+2)(s+3)} & \frac{1}{s^2+2} & 0 \end{bmatrix}, \qquad P = \begin{bmatrix} \frac{s+4}{s+1} & 0 & 0 \\ 0 & \frac{1}{s^2+2} & 0 \\ 0 & 0 & \frac{s+6}{s+3} \end{bmatrix}$$

$$G = \begin{bmatrix} -\frac{(s+4)(s^3+4s^2+s+5)}{(s+1)(s^2+s+5)} & -\frac{(s+2)(s+4)}{(s+1)(s^2+s+5)} & -\frac{(s+2)(s+6)(s^2+2)}{(s+1)(s^2+s+5)} \\ -\frac{(s+4)(s+5)(s^2+2)}{(s+2)(s^2+s+5)} & -\frac{(s+4)}{(s^2+s+5)} & -\frac{(s+6)(s^2+2)}{(s^2+s+5)} \\ -\frac{(s+4)^2(s+5)(s^2+2)}{(s+2)(s+3)(s^2+s+5)} & -\frac{(s+4)(s^3+5s^2+3s+13)}{(s+3)(s^2+2)(s^2+s+5)} & -\frac{(s+4)(s+6)(s^2+2)}{(s+3)(s^2+s+5)} \end{bmatrix}$$

Need to know $p^2 - p = 6$ to identify

- Abstraction with $S = \{1,2\}$

$$Q_S = \begin{bmatrix} 0 & \frac{(s+2)}{(s+1)} \\ \frac{(s+1)(s+5)(s^2+2)}{(s+2)(s^3+4s^2+s+5)} & 0 \end{bmatrix},$$

$$P_S = \begin{bmatrix} \frac{(s+4)}{(s+1)} & 0 & 0 \\ 0 & \frac{(s+4)}{(s^3+4s^2+s+5)} & \frac{(s+6)(s^2+2)}{(s^3+4s^2+s+5)} \end{bmatrix}$$

$$G_S = \begin{bmatrix} -\frac{(s+4)(s^3+4s^2+s+5)}{(s+1)(s^2+s+5)} & -\frac{(s+2)(s+4)}{(s+1)(s^2+s+5)} & -\frac{(s+2)(s+6)(s^2+2)}{(s+1)(s^2+s+5)} \\ -\frac{(s+4)(s+5)(s^2+2)}{(s+2)(s^2+s+5)} & -\frac{(s+4)}{(s^2+s+5)} & -\frac{(s+6)(s^2+2)}{(s^2+s+5)} \end{bmatrix}$$

Need to know $|S|^2 - |S| = 2$ to identify

Abstraction and Reconstruction Example

- Base Network

$$Q = \begin{bmatrix} 0 & \frac{s+2}{s+1} & 0 \\ 0 & 0 & \frac{s+3}{s+4} \\ \frac{(s+1)(s+5)}{(s+2)(s+3)} & \frac{1}{s^2+2} & 0 \end{bmatrix}, \quad P = \begin{bmatrix} \frac{s+4}{s+1} & 0 & 0 \\ 0 & \frac{1}{s^2+2} & 0 \\ 0 & 0 & \frac{s+6}{s+3} \end{bmatrix}$$

$$G = \begin{bmatrix} -\frac{(s+4)(s^3+4s^2+s+5)}{(s+1)(s^2+s+5)} & -\frac{(s+2)(s+4)}{(s+1)(s^2+s+5)} & -\frac{(s+2)(s+6)(s^2+2)}{(s+1)(s^2+s+5)} \\ -\frac{(s+4)(s+5)(s^2+2)}{(s+2)(s^2+s+5)} & -\frac{(s+4)}{(s^2+s+5)} & -\frac{(s+6)(s^2+2)}{(s^2+s+5)} \\ -\frac{(s+4)^2(s+5)(s^2+2)}{(s+2)(s+3)(s^2+s+5)} & -\frac{(s+4)(s^3+5s^2+3s+13)}{(s+3)(s^2+2)(s^2+s+5)} & -\frac{(s+4)(s+6)(s^2+2)}{(s+3)(s^2+s+5)} \end{bmatrix}$$

Ill-Posed

- Abstraction with $S = \{1,2\}$

$$Q_S = \begin{bmatrix} 0 & \frac{(s+2)}{(s+1)} \\ \frac{(s+1)(s+5)(s^2+2)}{(s+2)(s^3+4s^2+s+5)} & 0 \end{bmatrix},$$

$$P_S = \begin{bmatrix} \frac{(s+4)}{(s+1)} & 0 & 0 \\ 0 & \frac{(s+4)}{(s^3+4s^2+s+5)} & \frac{(s+6)(s^2+2)}{(s^3+4s^2+s+5)} \end{bmatrix}$$

$$G_S = \begin{bmatrix} -\frac{(s+4)(s^3+4s^2+s+5)}{(s+1)(s^2+s+5)} & -\frac{(s+2)(s+4)}{(s+1)(s^2+s+5)} & -\frac{(s+2)(s+6)(s^2+2)}{(s+1)(s^2+s+5)} \\ -\frac{(s+4)(s+5)(s^2+2)}{(s+2)(s^2+s+5)} & -\frac{(s+4)}{(s^2+s+5)} & -\frac{(s+6)(s^2+2)}{(s^2+s+5)} \end{bmatrix}$$

Representable and Ill-Posed

Network Reconstruction Example – Insights

1. DSF network reconstruction procedure works whether the network is well-posed, ill-posed, proper, improper, etc.
   - Well-posedness results ensure reconstructed network is sensible
2. Representable abstractions of ill-posed networks are ill-posed

Conclusions

Conclusions

- DSFs (Q, P) and DNFS model causal networks
- Abstractions are DSFs (DNFs) that preserve dynamics and causal dependencies
- Abstractions reduce the knowledge required to identify (reconstruct) a DSF (DNF)
- A DSF (DNF) is well-posed if and only if $(I - Q)$ has a proper inverse
- Given that an abstraction is representable (proper), it is well-posed if and only if the DSF (DNF) is well-posed

Questions?