Technical Notes and Correspondence

Necessary and Sufficient Conditions for Dynamical Structure Reconstruction of LTI Networks

Jorge Gonçalves and Sean Warnick

Abstract—**This paper formulates and solves the network reconstruction problem for linear time-invariant systems. The problem is motivated from a variety of disciplines, but it has recently received considerable attention from the systems biology community in the study of chemical reaction networks. Here, we demonstrate that even when a transfer function can be identified perfectly from input–output data, not even Boolean reconstruction is possible, in general, without more information about the system. We then completely characterize this additional information that is essential for dynamical reconstruction without appeal to ad-hoc assumptions about the network, such as sparsity or minimality.**

Index Terms—**Network reconstruction, networked systems, systems biology.**

I. Introduction

This paper explores the feasibility of obtaining a system's network structure information from input–output data. Although there are many situations where an understanding of network structure could be useful, a particular focus on this problem has grown most recently in the systems biology community [1], [2], [4], [6], [9], [11], [13]–[15]. There, researchers consider chemical reaction networks characterized by a large number of interacting species, and often the scientific interest is in understanding how different species impact each other. Frequently, even a simple understanding of these relationships, such as knowing that one species "promotes" or "inhibits" another, or even the mere presence of a causal interaction between species (the "Boolean" network structure), is sufficient for significant scientific progress.

While it has never been made precise, the difficulty with the problem of network reconstruction has been implicit in the work of various researchers who introduce additional assumptions to specify structure from data. Without characterizing exactly what additional information, if any, is necessary to infer the structure from data, these approaches suggest assumptions that are sufficient to enable input–output data to uniquely characterize network structure; the structure thus identified is then argued to be likely or reasonable in some sense.

The most common such assumption is the sparsity assumption; that is, that the sparsest network structure capable of producing the observed input–output behavior is a likely candidate for the actual system [8], [12]. In biology, one motivation for such an assumption is that the nonesential interaction between chemical species would be inefficient, and evolution would thus have favored more efficient (sparse) structures. While such reasoning is appealing, we demonstrate here that any input–output behavior, characterized by a transfer function, can be realized by a system that has a completely decoupled network structure. Since other network structures do exist in nature, additional assumptions are required that demand similar justification if one is to employ this approach to network reconstruction.

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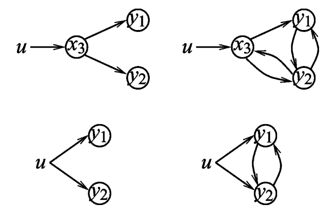

Fig. 1. Same transfer function yields two different minimal realizations and two different network structures: a decoupled internal structure (left) and coupled internal structure (right). Note that the complete realization contains hidden states (top), while the network structure only considers the causal relationship between measured states (bottom).

Our approach to the problem is different. Rather than hypothesize sufficient conditions that may or may not make sense in a particular application domain, we investigate precisely what information is essential to recover network structure from input–output data, regardless of the application domain. We find that even for linear time-invariant systems, and even assuming that the system transfer function can be perfectly recovered from input–output data, not even the Boolean network structure of the system can be obtained without more information unless the system has a single output, thereby trivializing the reconstruction problem. The following example demonstrates that even when the order of the system is known, and even if the system is known to be minimal, the Boolean structure of the system may still be unclear.

1) Example 1: Consider a system with the following transfer function:

$$G = \frac{1}{s+3}\begin{bmatrix}\frac{1}{s+1}\\ \frac{1}{s+2}\end{bmatrix}.$$

It can be shown that this transfer function is consistent with two systems with very different internal structures, given by

$$A_1 = \begin{bmatrix}-1 & 0 & 1\\ 0 & -2 & 1\\ 0 & 0 & -3\end{bmatrix}, \quad A_2 = \begin{bmatrix}-2 & -1 & 1\\ -1 & -3 & 1\\ 0 & -1 & -1\end{bmatrix}$$

$B_1 = B_2 = [0 \quad 0 \quad 1]'$, and $C_1 = C_2 = [I \quad 0]$ are 2×3 matrices. The networks in Fig. 1 correspond to each of the indicated realizations of G. Note that both realizations are minimal.

Such a result can appear to be very negative to scientists working on recovering even a very simplified understanding of the network structure of the nonlinear systems that arise in chemical reaction networks. Certainly if the network structure cannot be recovered from LTI systems, one should not expect the problem to get any easier for nonlinear systems. Nevertheless, we go on to precisely characterize the additional information necessary to recover the network structure in LTI systems, and we demonstrate the nature of experiments that enable network reconstruction without appeal to ad-hoc assumptions. For biochemists, these types of experiments are readily available and create a very encouraging roadmap for network reconstruction.

II. Dynamical Structure

Consider a system

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

that can be partitioned as

$$\begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \tag{1a}$$

$$y = [I \quad 0] \begin{bmatrix} y \\ z \end{bmatrix} \tag{1b}$$

where $x = [y' \quad z']' \in \mathbb{R}^n$ is the full state vector, $y \in \mathbb{R}^p$ is a partial measurement of the state, z are the $n-p$ "hidden" states, and $u \in \mathbb{R}^m$ is the control input. In this paper, we restrict our attention to situations where output measurements constitute partial state information (i.e., $p < n$). Note that in many applications the most sensible description of the system is in terms of the measured outputs as states, although rarely can we measure all states of the system. Also, we consider only systems with full rank transfer functions that do not have entire rows or columns of zeroes, since such "disconnected" systems are somewhat pathological and only serve to complicate the exposition without fundamentally altering the conclusions of this paper.

The complete network structure of this system (1) is a graph characterizing the casual relationships between the variables of interest (i.e., the inputs and states). This structure can be easily obtained from A and B. Nevertheless, when only partial state measurements are available, we may desire a characterization of the causal relationships between inputs and the measured states while suppressing structural information of the hidden states. To achieve this, we introduce dynamical structure functions as a representation of an LTI system that encodes structural information at the resolution of the measurements [5], where the resolution refers to the percentage of states that are measured p/n. If full state information is available, then the dynamical structure function of the system is equivalent to the full state-space description. If partial state information is available, then the dynamical structure function encodes the network structure of the system only as relevant to the measured states.

To make these ideas precise, consider the Laplace transform of the signals in (1), yielding

$$\begin{bmatrix} sY \\ sZ \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} Y \\ Z \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U \tag{2}$$

where Y, Z, and U are the Laplace transforms of y, z, and u, respectively. Solving for Z, gives

$$Z = (sI - A_{22})^{-1} A_{21} Y + (sI - A_{22})^{-1} B_2 U.$$

Substituting into (2) then yields

$$sY = WY + VU \tag{3}$$

where $W = A_{11} + A_{12}(sI - A_{22})^{-1} A_{21}$ and $V = A_{12}(sI - A_{22})^{-1} B_2 + B_1$. Let D be a matrix with the diagonal term of W (i.e., $D = \mathrm{diag}(W_{11}, W_{22}, \ldots, W_{pp})$). Then

$$(sI - D)Y = (W - D)Y + VU.$$

Note that $W - D$ is a matrix with zeroes on its diagonal. We then have

$$Y = QY + PU \tag{4}$$

where

$$Q = (sI - D)^{-1}(W - D) \tag{5}$$

and

$$P = (sI - D)^{-1} V. \tag{6}$$

The matrix Q is a matrix of transfer functions from Y_i to Y_j, $i \neq j$, relating each measured signal to all other measured signals. Note that Q is zero on the diagonal. Likewise, P is a matrix of transfer functions from each input to each output without depending on any additional measured state Y_i.

Definition 1: Given the system (1), we define the dynamical structure function of the system to be (Q, P), where Q and P are the internal structure and control structure, respectively, and given as in (5) and (6).

The dynamical structure function describes the network structure of the system (1) in the sense that the matrix Q can be interpreted as the weighted adjacency matrix of a directed graph indicating the causal relationships between measured states. The weights on edges of this graph are transfer functions between the relevant variables. Note that these transfer functions, the elements of Q and P, are strictly proper, thus preserving the interpretation of a causal relationship between variables. These relationships are indirect, in the sense that they may proceed through hidden states, but they are direct in the sense that the relationship cannot depend on any other measured state. Likewise, the matrix P can be interpreted as the weighted adjacency matrix of a directed graph, indicating the causal relationships from inputs to measured states. Note that the Boolean structure of the system is reflected in the zero structure of the matrices (Q, P), while the dynamical structure refers to the complete set of matrix functions.

We then see that just like a transfer function, a system's dynamical structure is uniquely specified. This implies that the state-space description of a system completely specifies the dynamical and Boolean structure of the system at every resolution level of measurements. We next consider the situation where the state-space description of the system is not known, and only the transfer function of the system is available.

Definition 2: A dynamical structure function (Q, P) is said to be consistent with a particular transfer function G, if a realization of G exists, of some order, and of the form (1), such that (Q, P) are specified by (5) and (6).

Definition 3: Consider a system characterized by a transfer function G. The dynamical structure of the system can be reconstructed if there is only one admissible dynamical structure function (Q, P) that is consistent with G. Likewise, the Boolean structure of the system can be reconstructed if all admissible dynamical structure functions that are consistent with G have the same Boolean structure. Here, "admissible" refers to the entries of Q and P being strictly proper rational functions, that Q is zero on the diagonal, and that (Q, P) satisfy any additional constraints, if any, that characterize the information about the system that is available *a priori*.

This paper explores necessary and sufficient conditions for the reconstruction of the network structure of a linear time-invariant dynamical system. The key property enabling dynamical and Boolean reconstruction is the uniqueness of a structure consistent with G, implying that it can be derived from the transfer function and, thus, by extension from input–output data.

III. Network Reconstruction

The network reconstruction problem is a question about the equivalence of different representations of a dynamical system. In particular, reconstruction concerns the equivalence between a system's transfer function and its dynamical structure function. This makes it similar to the realization problem, which is concerned with the equivalence between a system's transfer function and its state-space description. Note that the reconstruction problem, however, is less ambitious than the realization problem.

Current approaches to network reconstruction tend to focus on choosing a particular realization of a given transfer function, such as the sparsest realization [8], [12] or most parsimonious realization [3]. The critical feature of these approaches then becomes the justification of their chosen objective function for the application at hand: How do we know that the "true" realization optimizes sparsity, or parsimony, or any other metric that singles out a particular realization from the many realizations of a given transfer function? By solving a realization problem, all of these approaches fix the structure on the hidden states as well as the measured states of the system. Arguments must then be made that this structure—even on the hidden states—is correct.

The approach here is different. By explicitly formulating the reconstruction problem, we attempt only to find the structure between measured states, while leaving the structure between hidden states unspecified. This makes the reconstruction problem considerably less ambitious than the realization problem in situations where the number of hidden states is potentially large.

We begin by characterizing the situations when reconstruction is possible, knowing only the transfer function of a system. We consider dynamical and Boolean reconstruction, and show that neither is possible except in the degenerate case where there is only a single measurement $p = 1$, thus trivializing the reconstruction problem. We then consider what "extra information" would be necessary and sufficient to reconstruct the dynamical structure of a system with a known transfer function.

A. Reconstruction Without Extra Information

When only input–output data and no other "physical" understanding or partial information about the system are available, network reconstruction begins by solving a system identification problem. There are many important issues associated with the identification of a transfer function that best explains the data. These include the choice of a suitable model class, including the choice of system order; choice of a suitable metric to characterize what is meant by a "best" explanation of the data, methods for dealing with initial conditions and finite data records, techniques for handling noisy data, etc. These issues are the subject of current research and are discussed at length in the system identification literature; see, for example, [10] and references therein.

Nevertheless, in this paper, we assume that the identification problem has already been solved in the sense that the transfer function of the system from controlled inputs to measured outputs G has been correctly identified. In this section, we further assume that no additional information about the system is available. We begin by characterizing the relationship between a system's dynamical structure function and its transfer function.

Lemma 1: A dynamical structure function (Q, P) is consistent with a transfer function G if and only if the following relationship holds:

$$[\,G' \quad I\,]\begin{bmatrix} Q' \\ P' \end{bmatrix} = G' \tag{7}$$

where $'$ indicates the conjugate transpose.

Proof: Sufficiency follows by substituting $Y = GU$ into (4), yielding $GU = (QG + P)U$. Thus, $G = QG + P$, and (7) holds. Necessity likewise follows by construction. Equation (7) can be written as $(I - Q)G = P$ which means that $(I - Q)GU = PU$, for any vector U, or $Y = QY + PU$ since $Y = GU$. Multiplying both sides by s gives $sY = sQY + sPU$, which can be compared to (3), giving

$$A_{12}\,(sI - A_{22})^{-1}\,[\,A_{21} \quad B_2\,] + [\,A_{11} \quad B_1\,] = s\,[\,Q \quad P\,].$$

Matrices A and B can then be obtained by finding any state-space realization for $s\,[\,Q \quad P\,]$. ■

Lemma 1 demonstrates that a linear mapping from a system's dynamical structure function to its transfer function can be identified in terms of the transfer function. This leads immediately to the following observations.

Lemma 2: Given a $p \times m$ transfer function G, the operator $[\,G' \quad I\,]$ has the following properties:

1) has dimension $m \times (m + p)$;
2) has full-row rank;
3) has a nullspace of dimension p.

Lemma 3: Given a $p \times m$ transfer function $G(s)$ and for each value of s, every dynamical structure $[\,\tilde{Q} \quad \tilde{P}\,]'$ in the nullspace of the operator $[\,G' \quad I\,]$ satisfies

$$\tilde{P} = -\tilde{Q}G. \tag{8}$$

Proof: This follows immediately from $[\,G' \quad I\,][\,\tilde{Q} \quad \tilde{P}\,]' = 0$. ■

Lemma 4: Given a transfer function G, the set $\mathscr{S}_G$ of all dynamical structure functions consistent with G can be parameterized by a $p \times p$ internal structure function $\tilde{Q}$ and is given by

$$\mathscr{S}_G = \left\{ (Q, P) : \begin{bmatrix} Q' \\ P' \end{bmatrix} = \begin{bmatrix} 0 \\ G' \end{bmatrix} + \begin{bmatrix} I \\ -G' \end{bmatrix} \tilde{Q}', \tilde{Q} \in \mathscr{Q} \right\}$$

where $\mathscr{Q}$ is the set of internal structure functions. Moreover, the set $\mathscr{S}_G$ has $p^2 - p$ degrees of freedom.

Proof: From Lemma 1, consistency with G is ensured by (7). Any element from $\mathscr{S}_G$ satisfies (7) since $[\,G' \quad I\,][\,0 \quad G\,]' + [\,G' \quad I\,][\,I - G\,]'\tilde{Q}' = G' + 0 = G'$. Conversely, if a pair (Q, P) satisfies (7), then for any Q, $P = (I - Q)G$. The pair (Q, P) can then be seen to belong to $\mathscr{S}_G$, with $\tilde{Q}$ given by Q. The degrees of freedom in $\mathscr{S}_G$ follows from the fact that the parameter $\tilde{Q}$ has $p^2 - p$ degrees of freedom, since its diagonal entries are zero to yield admissible internal structure functions Q. ■

This decomposition of dynamical structure functions that is consistent with a given transfer function reveals the solution to the reconstruction problem. Note that when $p = 1$, dynamical and Boolean reconstruction are trivial since the only admissible Q is $Q = 0$, thus fixing $P = G$. Aside from this rather pathological situation, we characterize the solution to the reconstruction problem with the following theorem.

Theorem 1 (Reconstruction From G): Given any $p \times m$ transfer function G, with $p > 1$ and no other information about the system, dynamical and Boolean reconstruction are not possible. Moreover, for any internal structure Q, there is a dynamical structure function (Q, P) that is consistent with G.

Proof: The result follows immediately from Lemma 4. For any $\tilde{Q}$ with zero diagonal and nonzero elements that are strictly proper rational functions, $(Q, P) = (\tilde{Q}, (I - \tilde{Q})G)$ is consistent with G; neither dynamical nor Boolean reconstruction is possible without more information. ■

In particular, this shows that a completely decoupled $(Q = 0)$ and a fully connected internal structure (among others) is consistent with any given G. This fact highlights the point that a selection criteria for reconstruction, such as sparsity, must be justified independently of the input–output data from the system.

B. Reconstruction With Extra Information

Theorem 1 makes it clear that not even Boolean reconstruction is possible from G, in any interesting cases, without additional information. We know from Lemma 4 that the set of all dynamical structure functions consistent with a given transfer function has $p^2 - p$ degrees of freedom. The role of additional information, then, is to add enough constraints to the set of admissible dynamical structure functions such that the intersection of admissible dynamical structure functions, with the set of dynamical structure functions $\mathscr{S}_G$ that are consistent with a given transfer function, contains a unique element (Q, P). The following theorem indicates when partial structure information is necessary and sufficient for dynamical structure reconstruction, where partial structure information refers to the knowledge of some of the elements of Q or P.

Theorem 2 (Reconstruction With Partial Structure): Given a $p \times m$ transfer function G, dynamical structure reconstruction is possible from partial structure information if and only if $p-1$ elements in each column of $[\,Q \quad P\,]'$ are known that uniquely specify the component of (Q, P) in the nullspace of $[\,G' \quad I\,]$.

Proof: The nullspace of $[\,G' \quad I\,]$ has dimension p. Since the diagonal of Q is zero, then the component of each column of $[\,Q \quad P\,]'$ in the nullspace of $[\,G' \quad I\,]$ only has $p-1$ degrees of freedom. Thus, the "if" part follows from the knowledge of that component. For the "only if" part, assume there are not $p-1$ known elements in at least one column of $[\,Q \quad P\,]'$ that uniquely specify the component of (Q, P) in the nullspace of $[\,G' \quad I\,]$. The null space then has at least one degree of freedom. Let (Q_1, P_1) satisfy (7) with all of the known components of Q and P. Let $(\tilde{Q}, \tilde{P})$ also satisfy $\tilde{P} = -\tilde{Q}G$ and, thus, this is an element of the null space of $[\,G' \quad I\,]$ and restricted to have zero entries on the known components of Q and P. For the extra degrees of freedom of $\tilde{P} = -\tilde{Q}G$, pick as many as necessary $Q_{ij} = (s+1)^{-k}$ to uniquely specify all other elements of $(\tilde{Q}, \tilde{P})$, where k is large enough so that all $(\tilde{Q}, \tilde{P})$ are strictly proper. Then, $(Q_2, P_2) = (Q_1, P_1) + (\tilde{Q}, \tilde{P})$ is also a solution of (7) and all of the elements in (Q_2, P_2) are strictly proper. We have then found another dynamical structure (i.e., another set of strictly proper $Q_1 \neq Q_2$ and $P_1 \neq P_2$) consistent with G. Thus, the dynamical structure cannot be reconstructed uniquely, which is a contradiction. ■

The importance of Theorem 2 is that it identifies exactly what information about a system's structure, beyond knowledge of its transfer function, must be obtained to recover the rest of its structure without appealing to *a priori* assumptions, such as sparsity, or parsimony, etc. This enables the design of experiments to precisely target the extra information needed for reconstruction.

In many situations, we have no information on the internal structure Q, but we may partially know P since we are desigining the mechanisms actuating the system. When there is precisely one more measured state than inputs, $m = p-1$, then we observe that knowing all of the $mp = p^2 - p$ elements of P is sufficient to fully recover Q since the conditions of Theorem 2 are met. When $m < p-1$, however, some knowledge of Q will be essential for reconstruction.

A special case of Theorem 2 occurs when $p = m$ and G is full rank. In this situation, we simply observe that knowing P is diagonal (i.e., that the $p^2 - p$ offdiagonal elements of P are zero) is sufficient for reconstruction. Moreover, we demonstrate a particularly simple formula for (Q, P) in the following corollary.

Corollary 1: If $m = p$, G is full rank, and there is no information about the internal structure of the system Q, then the dynamical structure can be reconstructed if each input controls a measured state independently (i.e., without loss of generality, the inputs can be numbered such that P is diagonal). Moreover, $H = G^{-1}$ characterizes the dynamical structure as follows:

$$Q_{ij} = -\frac{H_{ij}}{H_{ii}} \text{ and } P_{ii} = \frac{1}{H_{ii}}.$$

Proof: The result follows directly from Theorem 2, as P, which is diagonal, implies that $p-1$ entries of each row of P are known to be zero. The formula for (Q, P) can be derived from (7), or (rearranging) $(I-Q)G = P$. Multiplying on the right by $H = G^{-1}$ yields $I - Q = PH$. Since Q has zeroes on its diagonal and P is diagonal, we have $1 = P_{ii}H_{ii}$ or $P_{ii} = 1/H_{ii}$. Finally, we can now solve for $Q = I - PH$ and the result follows. ■

Finally, we observe that knowing that full state measurements are available (i.e., $p = n$), is equivalent to knowing the structure of the system. This is because the realization of G with $C = I$ is unique, thereby generating a unique structure (Q, P) consistent with G. Nevertheless, such a situation seems impractical for most applications of interest.

IV. Conclusion

This paper formulated the network reconstruction problem for linear time-invariant systems. We introduced dynamical structure functions as a new representation of LTI systems and used them to show how the reconstruction problem differs from the realization problem.

The main results demonstrate that the map from the dynamical structure to a system's transfer function is generally not injective. As a result, extra structural information is necessary to derive the structure from input–output information. For any given transfer function, we then characterize the extra structural information that is needed for dynamical reconstruction.

We conclude by observing that the extra structural information needed for the reconstruction of LTI networks can be readily obtained from a sequence of biologically inspired "gene knockdown," and "over-expression" experiments [7]. Nevertheless, such a situation seems impractical for most applications of interest.

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On Inexact LPV Control Design of Continuous–Time Polytopic Systems

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***Abstract*—Robust dynamic output design most often relies on nonconvex problems. This is precisely the case with linear parameter-varying (LPV) control when the measured varying parameters do not exactly fit the real ones. This paper provides a solution in the convex programming framework with the use of linear matrix inequality (LMI) solvers in the case of polytopic parameter dependence. A bounded uncertainty set is defined from the difference between the measured and the true parameters from which LMI stabilization and performance conditions are obtained after bounding. As a desired property, the proposed conditions reduce to the classical LPV ones whenever the true parameters are available for control synthesis.**

***Index Terms*—Linear matrix inequality (LMI), linear parameter-varying (LPV) control design, uncertain linear systems.**

I. Introduction

Gain scheduling for linear parameter-varying (LPV) polytopic systems is an interesting answer for practical situations where real-time measurements are available to tune the controller according to parameter variations. Such situations are commonly encountered in a number of real-world problems from areas, such as aeronautics, aerospace, mechanics, and electronic devices as can be seen in the short list of references [2], [5], [6], [7], [13], [15], [16], among others. This has motivated interest in analysis, identification, and control design of LPV systems. In most of the works published, the time-varying parameters measurement is implicitly assumed to be exact. This is an important assumption under which the dynamic output control synthesis problem can be performed though semidefinite programming and especially linear matrix inequality (LMI) techniques.

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A seemingly important question can be put on the effect of a likely situation where the measurements are only made up to a given precision. This interest can be reinforced by the fact that the fragility issues have been shown to be important in robust control design with LMI techniques [8]. The issue of nonfragile, that is, robust LPV control has been introduced in some contributions, such as [1] in the case of linear fractional dependence on the varying parameters and [4] for the one of polytopic dependence. It is worth mentioning that in both cases, the dynamic output feedback control synthesis has to be performed by the solution of bilinear matrix inequalities, requiring the use of relaxation algorithms from which the sure convergence is generally lost. This is also the case for dynamic output feedback control of LPV systems with partly measured parameters, that is, where only some of the parameters are measured and available for feedback [9]. A rank minimization with LMI constraints is proposed as an alternative to the solvability conditions expressed in terms of a set of linear matrix inequalities and an additional coupling constraint which destroys the convexity of the overall problem. Until now, convexity is only obtained for analysis purpose as was done in [10] where only sufficient upper bounds on the mismatch between the true parameters and the ones used in the observer are proposed to guarantee that the error is bounded.

This paper addresses the problem in the polytopic case and provides a solution though LMI at the expense of some conservativeness. As in [1], the approach is derived within the framework of the so-called "Lyapunov Shaping Paradigm" where a time-invariant Lyapunov function is used for stability assessment ensuring stability and performance for arbitrary parameter time variation. There is now a growing interest for the use of polynomially parameter-dependent Lyapunov functions but their efficiency is still (and certainly for a long time from now) restricted to the analysis problem. A numerical example illustrates nevertheless the effectiveness of the proposed results. It is worth mentioning that an important feature of the proposed method is the ability to generate the classical LPV design whenever the error on the time-varying parameter vanishes.

The notation is standard. Capital letters denote matrices, small letters denote vectors, and small Greek letters denote scalars. For matrices or vectors, (') indicates transpose. For symmetric matrices, $X > 0$ (≥ 0) indicates that X is positive definite (positive semidefinite). The trace function denoted as $\mathrm{tr}(X)$ is equal to the sum of the eigenvalues of X. The set of real numbers is $\mathbb{R}$. To ease the notation, the symbol ($\bullet$) indicates the symmetric blocks appearing in LMIs. For trajectories $\xi(t)$ defined for all $t \geq 0$, the quantity $\|\xi\|_2^2 = \int_0^\infty \xi(t)'\xi(t)dt$ is its squared norm.

II. Problem Statement

Let a continuous-time LPV system be given by the following state-space realization:

$$\dot{x}(t) = A(\rho(t))x(t) + B_w w(t) + B_u u(t) \quad (1)$$
$$z(t) = C_z x(t) + D_z u(t) \quad (2)$$
$$y(t) = C_y x(t) + D_y w(t) \quad (3)$$

where as usual $x(\cdot) \in \mathbb{R}^n$ is the state, $u(\cdot) \in \mathbb{R}^m$ is the control, $w(\cdot) \in \mathbb{R}^r$ is the exogenous perturbation, $z(\cdot) \in \mathbb{R}^p$ is the controlled output, and $y(\cdot) \in \mathbb{R}^q$ is the measured output. To ease the presentation, it is assumed that only the matrix $A(\cdot)$ is uncertain and for each $t \geq 0$, it is given by the convex combination

$$A(\rho(t)) = \sum_{i=1}^{N} \rho_i(t) A_i \quad (4)$$