

Structured Coprime Factorizations Description of Linear and Time-Invariant Networks

Șerban Sabău[‡], Cristian Oară^{*}, Sean Warnick[†] and Ali Jadbabaie[‡]

Abstract—In this paper we study state-space realizations of Linear and Time-Invariant (LTI) systems. Motivated by biochemical reaction networks, Gonçalves and Warnick have recently introduced the notion of a *Dynamical Structure Functions* (DSF), a particular factorization of the system's transfer function matrix that elucidates the interconnection structure in dependencies between manifest variables. We build onto this work by showing an intrinsic connection between a DSF and certain sparse left coprime factorizations. By establishing this link, we provide an interesting systems theoretic interpretation of sparsity patterns of coprime factors. In particular we show how the sparsity of these coprime factors allows for a given LTI system to be implemented as a network of LTI sub-systems. We examine possible applications in distributed control such as the design of a LTI controller that can be implemented over a network with a pre-specified topology.

I. INTRODUCTION

Distributed and decentralized control of LTI systems has been a topic of intense research focus in control theory for more than 40 years. Pioneering work includes that of Radner [1], who revealed the sufficient conditions under which the minimal quadratic cost for a linear system can be achieved by a linear controller. Ho and Chu [2], laid the foundation of team theory by introducing a general class of distributed structures, dubbed *partially nested*, for which they showed the optimal LQG controller to be linear. More recently in [11], [12], [13], [14] important advances were made for the case where the decentralized nature of the problem is modeled as sparsity constraints on the input-output operator (the transfer function matrix) of the controller. These types of constraints are equivalent with computing the output feedback control law while having access to only partial measurements. Quite different from this scenario, in this work we are studying the meaning of sparsity constraints on the left coprime factors of the controller, which is not noticeable on its transfer function. In

particular, we show how the sparsity of these coprime factors allows for the given LTI controller to be implemented over a LTI network with a pre-specified topology.

More recently, network reconstruction of biochemical reaction networks have motivated a careful investigation into the nature of systems and the many interpretations of *structure* or sparsity structure one may define [18]. In this work, a novel partial structure representation for Linear Time-Invariant (LTI) systems, called the *Dynamical Structure Function* (DSF) was introduced. The DSF was shown to be a factorization of a system's transfer function that represented the *open-loop causal dependencies among manifest variables*, an interpretation of system structure dubbed the *Signal Structure*.

A. Motivation and Scope of Work

In this paper we look at Dynamical Structure Functions from a control systems perspective. A long standing problem in control of LTI systems was synthesis of decentralized stabilizing controllers ([4]) which means imposing on the controller's transfer function matrix $K(s)$ to have a diagonal sparsity pattern. Quite different to the decentralized paradigm, the ultimate goal of our research would be a systematic method of designing controllers that can be implemented as a LTI network with a pre-specified topology. This is equivalent with computing a stabilizing controller $K(s)$ whose DSF $(Q(s), P(s))$ satisfies certain sparsity constraints [20]. So, instead of imposing sparsity constraints on the transfer function of the controller as it is the case in decentralized control, we are interested in imposing the sparsity constraints on the controller's DSF.

B. Contribution

The contribution of this paper is the establishment of the intrinsic connections between the DSFs and the left coprime factorizations of a given transfer function and to give a systems theoretic meaning to sparsity patterns of coprime factors using DSFs. The importance of this is twofold. First, this is the most common scenario in control engineering practice (*e.g.* manufacturing, chemical plants) that the given plant is made out of many interconnected sub-systems. The *structure* of this interconnection is captured by a DFS description of the plant [19] which in turn might translate to left coprime factorization of the plant that features certain sparsity patterns on its factors. This sparsity might be used for the synthesis of a controller to be implemented over a LTI network. Conversely, in many applications it is desired that the stabilizing controller be implemented in a distributed

[‡]Șerban Sabău is with the Electrical and Computer Engineering Dept., Stevens Institute of Technology. Ș. Sabău was supported by AFOSR MURI CHASE, ONR MURI HUNT and AFOSR Complex Networks Program. email: ssabau@stevens.edu

^{*}Cristian Oară is with the Faculty of the Automatic Control Dept., University "Politehnica" Bucharest. C. Oară was supported by a grant of the Romanian National Authority for Scientific Research, CNCS UEFISCDI, project number PN-II-ID-PCE-2011-3-0235. email: cristian.oara@acse.pub.ro

[†]Sean Warnick is with the Information and Decision Algorithms Laboratories and the Faculty of the Computer Science Department at Brigham Young University and he gratefully acknowledges the support of AFRL FA8750-09-2-0219. email: sean.warnick@gmail.com.

[‡]Ali Jadbabaie is with the Faculty of the Electrical and Systems Engineering Dept., University of Pennsylvania. A. Jadbabaie was supported by AFOSR MURI CHASE, ONR MURI HUNT and AFOSR Complex Networks Program. email: jadbabai@seas.upenn.edu.

manner, for instance as a LTI network with a pre-specified topology. This is equivalent to imposing certain sparsity constraints on the left coprime factorization of the controller (via the celebrated Youla parameterization). In order to fully exploit the power of the DSFs approach to tackle these types of problems, we find it useful to underline its links with the classical notions and results in control theory of LTI systems. We provide here a comprehensive exposition of the elemental connections between the Dynamical Structure Functions and the Coprime Factorizations of a given Linear Time-Invariant (LTI) system, thus opening the way between exploiting the structure of the plant via the DSF and employing the celebrated Youla parameterization for feedback output stabilization.

C. Outline of the Paper

In the second Section of the paper we give a brief outline of the theoretical concept of Dynamical Structure Functions as originally introduced in [18]. In the third Section, we show that while the DSF representation of a given LTI system $L(s)$ is in general never coprime, a closely related representation dubbed a viable (W, V) pair associated with $L(s)$ is always coprime. We also provide the class of all viable (W, V) pairs associated with a given $L(s)$. The fourth Section contains the main results of the paper and it makes a complete explanation of the natural connections between the DSFs and the viable (W, V) pairs associated with a given $L(s)$ and its left coprime factorizations. The last Section contains the conclusions and future research directions. In the Appendix A we have provided a short primer on realization theory for improper TFMs which is indispensable for the proofs of the main results. In Appendix B we have placed brief description of the proofs. Complete proofs have been omitted due to lack of space. For detailed proofs of all the results included in this paper, we refer to [23].

II. DYNAMICAL STRUCTURE FUNCTIONS

The main object of study here is a LTI system, which in the continuous-time case are described by the state equations

$$\dot{x}(t) = Ax(t) + Bu(t); \quad x(t_o) = x_o \quad (1a)$$

$$y(t) = Cx(t) + Du(t) \quad (1b)$$

where A, B, C, D are $n \times n$, $n \times m$, $p \times n$, $p \times m$ real matrices, respectively while n is also called *the order* of the realization. Given any n -dimensional state-space representation (1a), (1b) of a LTI system (A, B, C, D) , its input-output representation is given by the *Transfer Function Matrix* (TFM) which is the $p \times m$ matrix with real, rational functions entries denoted with

$$L(\lambda) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \stackrel{def}{=} D + C(\lambda I_n - A)^{-1}B, \quad (2)$$

Remark 2.1: Our results apply on both continuous or discrete time LTI systems, hence we assimilate the undetermined λ with the complex variables s or z appearing in

the Laplace or Z-transform, respectively, depending on the type of the system.

For elementary notions in linear systems theory, such as state equivalence, controllability, observability, detectability, we refer to [8], or any other standard text book on LTI systems.

By $\mathbb{R}^{p \times m}$ we denote the set of $p \times m$ real matrices and by $\mathbb{R}(\lambda)^{p \times m}$ we denote $p \times m$ transfer function matrices (matrices having entries real-rational functions).

This section contains a discussion based on reference [18] on the definition of the Dynamical Structure Functions associated with a LTI system. We start with the given system $L(\lambda)$ described by the following state equations, of order n :

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}u(t); \quad \tilde{x}(t_o) = \tilde{x}_o \quad (3a)$$

$$y(t) = \tilde{C}\tilde{x}(t) \quad (3b)$$

Assumption 2.2: (Regularity) We make the assumption that the \tilde{C} matrix from (3b) has full row rank (it is surjective).

We choose any matrix \bar{C} such that $T \stackrel{def}{=} \begin{bmatrix} \tilde{C} \\ \bar{C} \end{bmatrix}$ is nonsingular (note that such \bar{C} always exists because \tilde{C} has full row rank) and apply a state-equivalence transformation

$$x(t) = T\tilde{x}(t), \quad A = T\tilde{A}T^{-1}, \quad B = T\tilde{B}, \quad C = \tilde{C}T^{-1}. \quad (4)$$

on (3a),(3b) in order to get

$$\begin{bmatrix} y(t) \\ z(t) \end{bmatrix} = T\tilde{x}(t) \quad (5a)$$

$$\begin{bmatrix} \dot{y}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t); \quad (5b)$$

$$y(t) = \begin{bmatrix} I_p & O \end{bmatrix} \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} \quad (5c)$$

Assumption 2.3: (Observability) We can assume without any loss of generality that the pair (\tilde{C}, \tilde{A}) from (3a), (3b) or equivalently the pair (A_{12}, A_{22}) from (5b) are observable.

Remark 2.4: The argument that the observability assumption does not imply any loss of generality, is connected with the Leuenberger reduced order observer, and is proved in detail in [23].

Looking at the Laplace or Z-transform of the equation in (5b), we get

$$\begin{bmatrix} \lambda I_p - A_{11} & -A_{12} \\ -A_{21} & \lambda I_{n-p} - A_{22} \end{bmatrix} \begin{bmatrix} Y(\lambda) \\ Z(\lambda) \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U(\lambda) \quad (6)$$

By multiplying (6) from the left with the following factor $\Omega(\lambda)$

$$\Omega(\lambda) = \begin{bmatrix} I_p & A_{12}(\lambda I_{n-p} - A_{22})^{-1} \\ O & I_{n-p} \end{bmatrix} \quad (7)$$

(note that $\Omega(\lambda)$ is always invertible as a TFM) we get

$$\begin{bmatrix} \left((\lambda I_p - A_{11}) - A_{12}(\lambda I_{n-p} - A_{22})^{-1} A_{21} \right) & O \\ * & * \end{bmatrix} \times \begin{bmatrix} Y(\lambda) \\ Z(\lambda) \end{bmatrix} = \Omega(\lambda) \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U(\lambda), \quad (8)$$

where the $*$ denote entries whose exact expression is not needed now. Immediate calculations yield that the first block-row in (8) is equivalent with

$$\begin{aligned} \lambda Y(\lambda) &= \left(A_{11} + A_{12}(\lambda I_{n-p} - A_{22})^{-1} A_{21} \right) Y(\lambda) + \\ &+ \left(B_1 + A_{12}(\lambda I_{n-p} - A_{22})^{-1} B_2 \right) U(\lambda) \end{aligned} \quad (9)$$

and by making the notation

$$W(\lambda) \stackrel{def}{=} -A_{11} - A_{12}(\lambda I_{n-p} - A_{22})^{-1} A_{21} \quad (10a)$$

$$V(\lambda) \stackrel{def}{=} B_1 + A_{12}(\lambda I_{n-p} - A_{22})^{-1} B_2 \quad (10b)$$

we finally get the following equation which describes the relationship between manifest variables

$$\lambda Y(\lambda) = W(\lambda)Y(\lambda) + V(\lambda)U(\lambda). \quad (11)$$

Remark 2.5: Note that if $V(\lambda)$ is identically zero, while $W(\lambda)$ is a constant matrix having the sparsity of a graph's Laplacian, then (11) becomes the free evolution equation $\lambda Y(\lambda) = WY(\lambda)$. These types of equations have been extensively studied in cooperative control [15] to describe the dynamics of a large group of autonomous agents. Equation (11) can be looked at as a generalization of that model and will be studied here in a different context.

Since $L(\lambda)$ is the input-output operator from $U(\lambda)$ to $Y(\lambda)$, we can write equivalently that $L(\lambda) = (\lambda I_p - W(\lambda))^{-1} V(\lambda)$, which is exactly the (W, V) representation from [18, (3)/ pp.1671]. (Note that since $W(\lambda)$ is always proper it follows that $(\lambda I_p - W(\lambda))$ is always invertible as a TFM.) Next, let $D(\lambda)$ denote the TFM obtained by taking the diagonal entries of $W(\lambda)$, that is $D(\lambda) \stackrel{def}{=} \text{diag}\{W_{11}(\lambda), W_{22}(\lambda) \dots W_{pp}(\lambda)\}$. Then we can write $L(\lambda) = \left[(\lambda I_p - D(\lambda)) - (W(\lambda) - D(\lambda)) \right]^{-1} V(\lambda)$, or equivalently (note that $(W - D)$ has zeros on the diagonal entries)

$$\begin{aligned} L(\lambda) &= \left[I - (\lambda I_p - D(\lambda))^{-1} (W(\lambda) - D(\lambda)) \right]^{-1} \times \\ &\times (\lambda I_p - D(\lambda))^{-1} V(\lambda) \end{aligned} \quad (12)$$

and after introducing the notation

$$Q(\lambda) \stackrel{def}{=} (\lambda I_p - D(\lambda))^{-1} (W(\lambda) - D(\lambda)) \quad (13a)$$

$$P(\lambda) \stackrel{def}{=} (\lambda I_p - D(\lambda))^{-1} V(\lambda) \quad (13b)$$

we get that $L(\lambda) = (I_p - Q(\lambda))^{-1} P(\lambda)$ or equivalently that

$$Y(\lambda) = Q(\lambda)Y(\lambda) + P(\lambda)U(\lambda) \quad (14)$$

Remark 2.6: The splitting and the “extraction” of the diagonal in (13a) are made in order to make the $Q(\lambda)$ have the sparsity (and the meaning) of the *adjacency matrix* of the graph describing the causal relationships between the manifest variables $Y(\lambda)$. Consequently, $Q(\lambda)$ will always have zero entries on its diagonal.

Definition 2.7: [18, Definition 1] Given the state-space realization of (5b),(5c) of $L(\lambda)$ the Dynamical Structure Function of the system is defined to be the pair $(Q(\lambda), P(\lambda))$, where $Q(\lambda), P(\lambda)$ are given by (13a) and (13b) respectively.

Although the Dynamical Structure Function is uniquely specified by a state space realization of the form (5b) and (5c), it is not uniquely specified by a given TFM. Instead, just as a TFM, $L(\lambda)$, has many state realizations, we note that it also has many Dynamical Structure Functions that are *consistent* with it. This idea is made precise in the following:

Definition 2.8: [18, Definition 2] Given a TFM, $L(\lambda)$, any two TFMs $Q(\lambda) \in \mathbb{R}^{p \times p}(\lambda)$ and $P(\lambda) \in \mathbb{R}^{p \times m}(\lambda)$ with $Q(\lambda)$ having zero entries on its diagonal, is a Dynamical Structure Function that is *consistent* with $L(\lambda)$ if $L(\lambda) = (I_p - Q(\lambda))^{-1} P(\lambda)$, or equivalently

$$Y(\lambda) = Q(\lambda)Y(\lambda) + P(\lambda)U(\lambda) \quad (15)$$

Definition 2.9: Given the TFM $L(\lambda)$, we call a *viable* $(W(\lambda), V(\lambda))$ pair associated with $L(\lambda)$, any two TFMs $W(\lambda) \in \mathbb{R}^{p \times p}(\lambda)$ and $V(\lambda) \in \mathbb{R}^{p \times m}(\lambda)$, with $W(\lambda)$ having McMillan degree at most $(n - p)$ and such that

$$L(\lambda) = (\lambda I_p - W(\lambda))^{-1} V(\lambda). \quad (16)$$

Proposition 2.10: Given a TFM $L(\lambda)$ then for any given *viable* $(W(\lambda), V(\lambda))$ pair associated with $L(\lambda)$, there exists a *unique* DSF representation $(Q(\lambda), P(\lambda))$ of $L(\lambda)$ given by (13a) and (13b), where $D(\lambda) \stackrel{def}{=} \text{diag}\{W_{11}(\lambda), W_{22}(\lambda) \dots W_{pp}(\lambda)\}$ is uniquely determined by $W(\lambda)$.

Proof: The proof follows immediately from the very definitions (13a), (13b). ■

Remark 2.11: It is important to remark here that any *viable* $(W(\lambda), V(\lambda))$ pair has the same sparsity pattern with its subsequent DSF representation $(Q(\lambda), P(\lambda))$. For example $W(\lambda)$ is lower triangular if and only if $Q(\lambda)$ is lower triangular. Similarly, for instance $V(\lambda)$ is tridiagonal if and only if $P(\lambda)$ is tridiagonal.

III. MAIN RESULTS

The ultimate goal of this line of research would be computing controllers whose DSF has a certain structure. This would allow us for instance to compute controllers that can be implemented as a “ring” network (see Figure ??) or as a “line” network which is important for motion control of vehicles moving in a platoon formation. However, classical results in LTI systems control theory, such as the celebrated Youla parameterization (or its equivalent formulations) render the expression of the stabilizing controller as a stable

coprime factorization of its transfer function. As a first step towards employing Youla-like methods for the synthesis of controllers featuring structured DSF, we need to understand the connections between the stable left coprime factorizations (of a given stabilizing controller) and its DSF representation. We address this problem in this section.

A. Main Result

In this subsection, given a TFM $L(\lambda)$, we provide closed state-space formulas for the parameterization of the class of all viable pairs $(W(\lambda), V(\lambda))$ associated with $L(\lambda)$, for which $(\lambda I_p - W(\lambda), V(\lambda))$ is also a *left coprime factorization* of $L(\lambda)$.

An equivalent condition for $(\lambda I_p - W(\lambda), V(\lambda))$ to be left coprime is for the compound transfer function matrix

$$\begin{bmatrix} (\lambda I_p - W(\lambda)) & V(\lambda) \end{bmatrix} \quad (17)$$

to have no (finite or infinite) Smith zeros (see [3], [9], [10] for equivalent characterizations of left coprimeness). Coprimeness is especially important for output feedback stabilization, since classical results such as the celebrated Youla parameterization, require a coprime factorization of the plant while also rendering coprime factors of the stabilizing controllers.

Assumption 3.1: (Controllability) From this point onward we assume that the realization (3a),(3b) of $L(\lambda)$ is controllable.

Theorem 3.2: Given a TFM $L(\lambda)$ having a state-space realization (3a),(3b), we compute any equivalent realization (5b),(5c). The class of all *viable* $(W(\lambda), V(\lambda))$ pairs associated with $L(\lambda)$, for which $(\lambda I_p - W(\lambda), V(\lambda))$ is also a *left coprime factorization* of $L(\lambda)$ is given by

$$-W(\lambda) = \left[\frac{(A_{22} + KA_{12}) - \lambda I_{n-p}}{A_{12}} \mid \frac{A_{22}K + KA_{12}K - KA_{11} - A_{21}}{-A_{11} + A_{12}K} \right] \quad (18)$$

$$V(\lambda) = \left[\frac{(A_{22} + KA_{12}) - \lambda I_{n-p}}{A_{12}} \mid \frac{KB_1 + B_2}{B_1} \right] \quad (19)$$

where the K is any matrix in $\mathbb{R}^{(n-p) \times p}$ and $A_{11}, A_{12}, A_{21}, A_{22}, B_1, B_2$ are as in (5b),(5c).

Proof: See [23]. ■

Remark 3.3: We remark here the poles of both $W(\lambda)$ and $V(\lambda)$ can be allocated at will in the complex plane, by a suitable choice of the matrix K and the assumed observability of the pair (A_{12}, A_{22}) (Assumption 2.3).

Remark 3.4: We remark here that while any viable $(W(\lambda), V(\lambda))$ pair associated with a given $T(\lambda)$ makes out for a left coprime factorization $L(\lambda) = (\lambda I_p - W(\lambda))^{-1}V(\lambda)$, the DSF $L(\lambda) = (I_p - Q(\lambda))^{-1}P(\lambda)$ are in general never coprime (unless the plant is stable or diagonal). That is due to the fact that in general not all the unstable zeros of $(\lambda I_p - D(\lambda))$ cancel out when forming the products

in (13a), (13b) and the same unstable zeros will result in poles/zeros cancellations when forming the product $L(\lambda) = (I_p - Q(\lambda))^{-1}P(\lambda)$.

B. Getting from DSFs to Stable Left Coprime Factorizations

In this subsection we show that for any viable pair $(W(\lambda), V(\lambda))$ with both $W(\lambda)$ and $V(\lambda)$, respectively being stable, there exists a class of stable left coprime factorizations. Furthermore, there exists a class of stable left coprime factorizations that preserve the sparsity pattern of the original viable pair $(W(\lambda), V(\lambda))$.

Note that for any viable pair $(W(\lambda), V(\lambda))$ is an *improper* rational function and it has exactly p poles at infinity of multiplicity one, hence the $(\lambda I_p - W(\lambda))$ factor (the denominator of the factorization) is inherently *unstable* (in either continuous or discrete-time domains). We remind the reader that any the poles of both $W(\lambda)$ and $V(\lambda)$ can be allocated at will in the stability domain (Remark 3.3). In this subsection, we show how to get from viable pair $(W(\lambda), V(\lambda))$ of $L(\lambda)$ in which both factors $W(\lambda)$ and $V(\lambda)$ are stable, to a stable left coprime factorization $L(\lambda) = M^{-1}(\lambda)N(\lambda)$. We achieve this without altering any of the stable poles of $W(\lambda)$ and $V(\lambda)$ (which are the modes of $(A_{22} + KA_{12})$ in (18), (19)) and while at the same time keeping the McMillan degree to the minimum. The problem is to displace the p poles at infinity (of multiplicity one) from the $(\lambda I_p - W(\lambda))$ factor. To this end we will use the *Basic Pole Displacement Result* from [10, Theorem 3.1] that shows that this can be achieved by premultiplication with an adequately chosen invertible factor $\Theta(\lambda)$ such that when forming the product $\Theta(\lambda)(\lambda I_p - W(\lambda))$ all the p poles at infinity of the factor $(\lambda I_p - W(\lambda))$ cancel out. Here follows the precise statement:

Lemma 3.5: Given a viable pair $(\lambda I_p - W(\lambda), V(\lambda))$ of $L(\lambda)$ then for any

$$\Theta(\lambda) \stackrel{def}{=} \left[\frac{A_x - \lambda I_p}{T_5} \mid \frac{T_4}{O} \right] \quad (20)$$

with A_x, T_4, T_5 arbitrarily chosen such that A_x has only stable eigenvalues and both T_4, T_5 are invertible, it follows that

$$\begin{bmatrix} M(\lambda) & N(\lambda) \end{bmatrix} \stackrel{def}{=} \Theta(\lambda) \begin{bmatrix} (\lambda I_p - W(\lambda)) & V(\lambda) \end{bmatrix} \quad (21)$$

is a *stable* left coprime factorization $L(\lambda) = M^{-1}(\lambda)N(\lambda)$. Furthermore,

$$\begin{bmatrix} M(\lambda) & N(\lambda) \end{bmatrix} = \left[\frac{A_x - \lambda I_{n-p}}{T_4^{-1}} \mid \frac{T_4 A_{12}}{O} \right] \left[\frac{(A_x T_4 - T_4 A_{11} + T_4 A_{12} K)}{(A_{22} K + K A_{12} K - K A_{11} - A_{21})} \mid \frac{T_4 B_1}{O} \right] \quad (22)$$

hence all the modes in $(A_{22} + KA_{12})$ (which are the original stable poles of $W(\lambda)$ and $V(\lambda)$) are preserved in the $M(\lambda)$ and $N(\lambda)$ factors.

Proof: See [23]. ■

Remark 3.6: We remark that for any diagonal A_x having only stable eigenvalues $\Theta(\lambda) = (\lambda I_p - A_x)^{-1}$ yields a stable left coprime factorization of $L(\lambda)$ that preserves the *sparsity structure* of the initial viable $(\lambda I_p - W(\lambda), V(\lambda))$ pair.

C. Connections with the Nett & Jacobson Formulas [16]

In this subsection, we are interested in connecting the expression from (22) for the pair $(M(\lambda), N(\lambda))$ to the classical result of state-space derivation of left coprime factorizations of a given plant originally presented in [16] (and generalized in [17]).

Proposition 3.7: [16], [17] Let $L(\lambda)$ be an arbitrary $m \times p$ TFM and Ω a domain in \mathbb{C} . The class of *all* left coprime factorizations of $L(\lambda)$ over Ω , $T(\lambda) = M^{-1}(\lambda)N(\lambda)$, is given by

$$\begin{bmatrix} M(\lambda) & N(\lambda) \end{bmatrix} = U^{-1} \left[\begin{array}{c|cc} (A - FC) - \lambda I & -F & B \\ \hline C & I & O \end{array} \right], \quad (23)$$

where A, B, C, F and U are real matrices accordingly dimensioned such that

- i)** U is any $p \times p$ invertible matrix,
- ii)** F is any feedback matrix that allocates the observable modes of the (C, A) pair to Ω ,
- iii)** $L(\lambda) = \left[\begin{array}{c|c} A - \lambda I & B \\ \hline C & O \end{array} \right]$ is a stabilizable realization.

Due to Assumption 3.1, we have to replace the stabilizability from point **iii)** with a controllability assumption. We start off with $L(\lambda)$ given by the equations (3a),(3b)

$$L(\lambda) = \left[\begin{array}{cc|c} A_{11} - \lambda I_p & A_{12} & B_1 \\ A_{21} & A_{22} - \lambda I_{n-p} & B_2 \\ \hline I & O & O \end{array} \right] \quad (24)$$

and we want to retrieve (22) by using the parameterization in Proposition 3.7. First apply a state-equivalence $T = \begin{bmatrix} T_4 & O \\ K & I \end{bmatrix}$ in order to get

$$L(\lambda) = \left[\begin{array}{c|cc} \frac{T_4(A_{11} - A_{12}K)T_4^{-1} - \lambda I_p}{T_4^{-1}} & & \\ \hline T_4 A_{12} & T_4 B_1 & \\ KA_{12} + A_{22} - \lambda I_{n-p} & KB_1 + B_2 & \\ \hline O & O & \end{array} \right] \quad (25)$$

Next, we only need to identify the F feedback matrix from point **ii)** of Proposition 3.7, which in this case is proven to be given by

$$F = \left[\begin{array}{c} (T_4^{-1}A_x T_4) - A_{11} + A_{12}K \\ -K(T_4^{-1}A_x T_4) + A_{22}K - A_{21} \end{array} \right] \quad (26)$$

To check, simply plug (26) in (23) for the realization (25) of $L(\lambda)$.

D. Getting from the Stable Left Coprime Factorization to the DSFs

In this subsection we show that for *almost* every stable left coprime factorization of a given LTI system, there is an associated a *unique* viable $(W(\lambda), V(\lambda))$ pair and consequently (via Remark 2.10) a unique DSF representation $(Q(\lambda), P(\lambda))$. The key role in establishing this one to one correspondence is played by a non-symmetric Riccati equation, whose solution existence is a generic property. This result is meaningful, since for controller synthesis while we are interested in the DSF of the controller, in general we only have access to a stable left coprime of the controller.

We start with a given stable left coprime factorization (23) for $L(\lambda)$ having an order n realization

$$\begin{bmatrix} M(\lambda) & N(\lambda) \end{bmatrix} = U^{-1} \left[\begin{array}{c|cc} (A - FC) - \lambda I & -F & B \\ \hline C & I & O \end{array} \right] \quad (27)$$

to which we apply a type (4) state-equivalence transformation with $T \in \mathbb{R}^{n \times n}$ such that $CT^{-1} = \begin{bmatrix} I_p & O \end{bmatrix}$. Note that such a T always exists because of Assumption 2.2. It follows that (27) takes the form

$$\begin{bmatrix} M(\lambda) & N(\lambda) \end{bmatrix} = \left[\begin{array}{cc|cc} A_{11} + F_1 - \lambda I_p & A_{12} & F_1 & B_1 \\ A_{21} + F_2 & A_{22} - \lambda I_{n-p} & F_2 & B_2 \\ \hline I_p & O & I_p & O \end{array} \right] \quad (28)$$

and denote

$$A^+ \stackrel{def}{=} \left[\begin{array}{cc} A_{11} + F_1 & -A_{12} \\ -(A_{21} + F_2) & A_{22} \end{array} \right] \quad (29)$$

The solution of the following nonsymmetric algebraic Riccati matrix equation is paramount to the main result of this subsection, since it underlines the one to one correspondence between (28) and its *unique* associated viable $(W(\lambda), V(\lambda))$ pair.

Proposition 3.8: The nonsymmetric algebraic Riccati matrix equation

$$K(A_{11} + F_1) - A_{22}K - KA_{12}K + (A_{21} + F_2) = O \quad (30)$$

has a stabilizing solution K (i.e. $(A_{11} + F_1 - A_{12}K)$ is stable) if and only if the A^+ matrix from (29) has a stable invariant subspace of dimension p with basis matrix

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \quad (31)$$

having V_1 invertible (i.e. disconjugate). In this case $K = V_1^{-1}V_2$ and it is the *unique* solution of (30).

Proof: It follows from [22]. ■

Remark 3.9: Since in our case A^+ is stable, all its invariant subspaces are actually stable (including the whole space). Therefore, the Riccati equation has a stabilizing solution if and only if the matrix A^+ has an invariant subspace of dimension p which is disconjugate. Hence, if for example

A^+ has only simple eigenvalues, the Riccati equation always has a solution (we can always select p eigenvectors (from the n eigenvectors) to form a disconjugate invariant subspace). In this case, all we have to do is to order the eigenvalues in a Schur form such that the corresponding invariant subspace has V_1 invertible. Although this is a *generic property*, when having Jordan blocks of dimension greater than one it might happen that the matrix A^+ has no disconjugate invariant subspace of appropriate dimension p , and therefore the Riccati equation has no solution (stable or otherwise).

Theorem 3.10: Given any stable left coprime factorization $L(\lambda) = M^{-1}(\lambda)N(\lambda)$ and its state-space realization (28), let K be the solution of the nonsymmetric algebraic Riccati equation (30) and denote $A_x \stackrel{def}{=} (F_1 + A_{11} - A_{12}K)$. Then, a state-space realization for $\begin{bmatrix} M(\lambda) & N(\lambda) \end{bmatrix}$ is given by

$$\begin{bmatrix} M(\lambda) & N(\lambda) \end{bmatrix} = \left[\begin{array}{cc|c} A_x - \lambda I_{n-p} & A_{12} & \\ \hline O & A_{22} + KA_{12} - \lambda I_p & \\ \hline I & O & \end{array} \right] \left(\begin{array}{cc|c} (A_x - A_{11} + A_{12}K) & B_1 & \\ \hline (A_{22}K + KA_{12}K - KA_{11} - A_{21}) & KB_1 + B_2 & \\ \hline I & O & \end{array} \right) \quad (32)$$

Furthermore, from (32) we can recover the exact expression of the subsequent viable $(W(\lambda), V(\lambda))$ pair associated with $L(\lambda)$, where $W(\lambda)$ and $V(\lambda)$ are given by (18) and (19), respectively.

Proof: For the proof, simply plug

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \stackrel{def}{=} \begin{bmatrix} A_x - A_{11} + A_{12}K \\ -KA_x + A_{22}K - A_{21} \end{bmatrix} \quad (33)$$

into the expression of (28) in order to obtain (32). The rest of the proof follows from Lemma 3.5, by taking T_4 to be equal with the identity matrix I_p . ■

Remark 3.11: We remark here that in general there is no correlation between the sparsity pattern of the stable left coprime (28) we start with and its associated viable $(W(\lambda), V(\lambda))$ pair produced in Theorem 3.10. That is to say that the converse of the observation made in Remark 3.6 is not valid. This poses additional problems for controller synthesis, since it might happen to encounter stable left coprime factorizations that have no particular sparsity pattern (are dense TFM) while their associated viable $(W(\lambda), V(\lambda))$ pair are sparse. This is due to the fact that in general, the A_x matrix in Theorem 3.10 can be a dense matrix. One way to circumvent this problem would be to use a carefully adapted version of Youla's parameterization in which the stable left coprime factorization to be replaced with a DSF description where both with $(W(\lambda), V(\lambda))$ factors are stable. This is the topic of our future investigation.

REFERENCES

- [1] R. Radner, "Team decision problems", *Ann. Math. Statist.*, vol. 33, pp. 857881, 1962.
- [2] Y.-C. Ho, K. C. Chu, "Team decision theory and information structures in optimal control problems Part I", *IEEE Trans. Autom. Control*, vol. AC-17, no. 1, pp. 1522, Jan. 1972.
- [3] H. H. Rosenbrock, *State-Space and Multivariable Theory*, Wiley, New York, 1970.
- [4] Wang, S.H., Davison, E.J. "On the stabilization of decentralized control systems", *IEEE Trans. on Automatic Control*, vol AC-18, no 5, October 1973, pp 473-478.
- [5] G. Verghese, P. Van Dooren, and T. Kailath, "Properties of the system matrix of a generalized state-space system", *Int. J. Control*, 30:235-243, 1979.
- [6] P. Van Dooren, "The generalized eigenstructure problem in linear system theory", *IEEE Trans. Autom. Control*, vol. 26, pp. 111-129, Jan. 1981.
- [7] G. Verghese, B. Lévy, and T. Kailath, "A generalized state-space for singular systems", *IEEE Trans. Autom. Control*, vol. 26, pp. 811-831, 1981.
- [8] Murray W. Wonham, "Linear Multivariable Control", *Springer-Verlag*, 1985.
- [9] M. Vidyasagar "Control System Synthesis: A Factorization Approach", *MIT Press*, Signal Processing, Optimization, and Control Series, 1985.
- [10] C. Oară, A. Varga, "Minimal Degree Factorization of Rational Matrices", *SIAM J. Matrix Anal. Appl.*, Vol. 21, No.1, 1999. (pp. 245-277)
- [11] P. Voulgaris, "Control of nested systems", in *Proceedings of the American Control Conference*, 2000, pp. 44424445.
- [12] P. Voulgaris, "A convex characterization of classes of problems in control with specific interaction and communication structures", in *Proceedings of the American Control Conference*, 2001, pp. 31283133.
- [13] M. C. Rotkowitz, S. Lall, "A characterization of convex problems in decentralized control", *IEEE Transactions on Automatic Control*, vol. 51, no. 2, pp. 274286, 2006.
- [14] P. Shah and P.A. Parrilo, "An optimal controller architecture for poset-causal systems", available on arxiv.org
- [15] A. Jadbabaie, J. Lin, and A. S. Morse "Coordination of groups of mobile autonomous agents using nearest neighbor rules" *IEEE Transactions on Automatic Control*, Vol. 48, No. 6, June 2003, pp. 988-1001.
- [16] C.N. Nett, C.A. Jacobson, M.J. Balas, "A connection between state-space and doubly coprime fractional representations", *IEEE Trans. Automat. Control* 29 (9) (1984) pp. 831832.
- [17] V. M. Lucic, "Doubly coprime factorization revisited", *IEEE Transactions on Automatic Control*, vol. 46, no. 3, pp. 457 459, 2001.
- [18] J. Gonçalves, S. Warnick "Necessary and Sufficient Conditions for Dynamical Structure Reconstruction of LTI Networks" *IEEE Trans. Autom. Control*, vol. 53, pp. 1670-1674, 2008.
- [19] E. Yeung, J. Goncalves, H. Sandberg and S. Warnick, "Mathematical Relationships Between Representations of Structure in Linear Interconnected Dynamical Systems", *American Control Conference*, San Francisco, 2011.
- [20] A. Rai and S. Warnick, "A Technique for Designing Stabilizing Distributed Controllers with Arbitrary Signal Structure Constraints," to appear in the *Proceedings of the European Control Conference*, Zurich, Switzerland, 2013.
- [21] C. Oară, Ş. Sabău "Minimal indices cancellation and rank revealing factorizations for rational matrix functions" *Linear Algebra and its Applications*, vol.431, pp. 17851814, 2009.
- [22] M. Jungers, C. Oara "Non-symmetric algebraic Riccati theory: a matrix pencil approach", *European Journal of Control*, Vol. 18 (1), pag 74-81, 2012.
- [23] Ş. Sabău, C. Oară, S. Warnick, A. Jadbabaie "Structured Coprime Factorizations Description of Linear and Time-Invariant Networks" Available on arxiv.org