2015 IEEE 54th Annual Conference on Decision and Control (CDC)
December 15-18, 2015. Osaka, Japan

Network Semantics of Dynamical Systems

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Abstract— **Dynamical systems enjoy a rich variety of mathematical representations, from interconnections of convolution operators or rational functions of a complex variable to systems of (possibly stochastic) differential or differential-algebraic equations. Although many of these representations can describe the same *behavior*, i.e. represent the same constraints on manifest variables, each one may characterize a different notion of *system structure*.**

This paper introduces a method for interpreting the semantics of different representations of a network system by exploring the set of realizations consistent with each. We then focus on signal structure, extending its definition, and demonstrate that its semantics differ from other network representations in important and useful ways. In particular, the information cost for identifying a system's signal structure from data can be considerably less than that needed for identifying a system's subsystem structure.

I. Introduction

Dynamics and structure are two of the most important properties of a dynamical system. Nevertheless, while a system's dynamics describe its behavior, i.e. how it constrains allowed combinations of measured or *manifest* variables, the notion of structure in a system is a property of its representation. Since a variety of representations can describe the same system as specified by a fixed behavior (e.g. the transfer function matrix of an LTI system as well as any of its state realizations), every system can be associated with a variety of internal structures.

In this work, we will consider different representations of the same system that are dynamically equivalent but structurally more or less detailed. A single structurally less detailed representation will then be consistent with a *set* of structurally more detailed representations, and this set is what defines the meaning, or *semantics*, of the less detailed representation.

This presence of multiple representations, and their corresponding network structures, associated with any dynamical system raises significant questions about any structural analyses performed on such systems. For example, which type of structure does a network identification algorithm discover, or which type of structure should characterize the design constraints in a distributed or networked control problem? Using the wrong type of structure for different problems can lead to misleading–and even incorrect–results.

This material is based on research sponsored by the Department of Homeland Security (DHS) Science and Technology Directorate, Homeland Security Advanced Research Projects Agency (HSARPA), Cyber Security Division (DHS S&T/HSARPA/CSD), LRBAA 12-07 via contract number HSHQDC-13-C-B0052. All material in this paper represents the position of the authors and not necessarily that of DHS.

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In the next section the notion of semantics for the transfer function is discussed in terms of state space representations. Then in Section III two partial structure representations are introduced: the subsystem structure and signal structure. The definition of the dynamical structure function, an LTI representation of the signal structure developed in [1], is extended to a more general class of systems in Section III-B. Finally, the notion of semantics for the partial structure representations are detailed in Section IV.

II. Motivating Example

We illustrate this method of characterizing semantics of a system representation with the following example, using an LTI system's transfer function (or, equivalently, its associated graph characterizing the system's *manifest structure*, see [2] for details) as a less detailed representation of the system, and its corresponding set of state space realizations as the set of models (among all possible state space models) that characterize the *semantics* of that particular transfer function representation of the system.

Example 1. *Consider the following* 2×2 *transfer function matrix:*

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+2} \end{bmatrix} \tag{1}$$

where s *is the Laplace variable. Notice that the structure, i.e. the sparsity pattern, of the transfer function is diagonal.*

One possible minimal realization with consistent dynamics is given by:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, D = 0 \tag{2}$$

where the A *matrix has the same diagonal structure as the original transfer function in (1). However, the following minimal realization is also consistent with the transfer function given in (1):*

$$A = \begin{bmatrix} -4 & 3 \\ -2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}, C = \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}, D = 0 \tag{3}$$

It is evident that the internal structure of the state space given by the sparsity pattern of the A *matrix now can be full, even though the transfer function is still diagonal.*

This example illustrates two important points:

1) The transfer function of a system contains very little information about the structure of any of its state space realizations.
2) The semantics of this transfer function in (1) restrict the set of associated state space realizations to structures where the net effects of a pathway from input i to output

j are zero when $i \neq j$, but it includes realizations with associated graphical structures that are fully connected and are not necessarily block diagonal.

III. Partial Structure Representations

Transfer functions and state space representations are some of the most common system representations in linear systems theory and are well-studied in the literature. We call the network structures associated with these representations the *manifest structure* and *complete computational structure*, respectively [2]. We now introduce network structures associated with two partial structure representations: the subsystem structure, which has a fairly rich history in the area of cooperative control and multi-agent systems [3], [4], [5], and signal structure, characterizing emerging representations such as dynamical structure functions or linear dynamical graphs [6], [7], [8], [9], [10], [11].

A. Subsystem Structure

Subsystem structure is intuitive, easy to understand, and exploits one of the great properties of systems theory, that the interconnection of systems yields another system. Since the internal structures of each subsystem can be obfuscated, decomposing large complex systems into relevant subsystems yields a (potentially hierarchical) view of the system that helps manage representational complexity.

Definition 1. *The linear **subsystem structure** dynamics for the i^{th} subsystem is given by*

$$\dot{x}_i \;=\; A_i x_i + \textstyle\sum_{j=1}^{n} W_{ij} x_j + B_i u_i \tag{4}$$

where $i = 1, ..., q$, $j \neq i$, $x_k \in \mathbb{R}^{n_k}$, $u_k \in \mathbb{R}^{m_k}$, and A_i, W_{ij}, and B_i are of appropriate dimension.

Subsystems only interact with other subsystems as established through the subsystem structure. However, certain problems might couple the subsystems through their objectives. Moreover, an interconnection of subsystems has, by definition, the critical feature that *states internal to one subsystem are not employed by other subsystems*.

B. Signal Structure

The signal structure characterizes the open-loop causal dependencies among manifest variables, as opposed to the manifest structure, which reflects structural information about the internal closed-loop behavior of the system (where "closed-loop" in this context means the net effect of internal components interacting to create the observed external dynamic behavior). For LTI systems the signal structure is consistent with a left co-prime factorization of the transfer function matrix [12].

The signal structure for LTI systems is characterized by an equation of the form:

$$Y = QY + PU. \tag{5}$$

where Y represent measured states, U measured inputs, and (Q, P) contain information about structural relationships. We call (Q, P) the system's *dynamical structure function* and let its *signal structure* be the graph associated with Q and P interpreted as adjacency matrices. Although a number of researchers develop system representations in this form [7], [13], [1], we will follow the development in [1] and [2].

The dynamical structure function of a system was originally defined in [1] for state space systems where $C = \begin{bmatrix} I & 0 \end{bmatrix}$ and $D = 0$. This was extended in [14], [15] to allow for general state space realizations; however, the transformation presented by the authors did not have some important properties, such as invariance to state permutations. The necessity of such a property is made evident in the following example:

Example 2. *Consider the following state space representation of a system:*

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & -1 \\ 2 & 0 & -3 \end{bmatrix} x + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} x \end{aligned} \tag{6}$$

The definition of the dynamical structure function from [14] yields

$$Q(s) = \begin{bmatrix} 0 & 0 \\ \frac{-2}{(s+2)(s+5)} & 0 \end{bmatrix},$$

$$P(s) = \begin{bmatrix} \frac{s+5}{s^2+4s+1} & 0 & \frac{s+2}{s^2+4s+1} \\ 0 & \frac{1}{s+2} & \frac{-1}{(s+2)(s+5)} \end{bmatrix} \tag{7}$$

Now consider a permutation on the system in (6) which renumbers the first and third states. The associated dynamical structure function, again using the definition in [14], is then given by

$$Q(s) = \begin{bmatrix} 0 & 0 \\ \frac{-s-1}{(s+2)^2} & 0 \end{bmatrix},$$

$$P(s) = \begin{bmatrix} \frac{s+5}{s^2+4s+1} & 0 & \frac{s+2}{s^2+4s+1} \\ \frac{1}{(s+2)^2} & \frac{1}{s+2} & 0 \end{bmatrix} \tag{8}$$

which is not consistent with the dynamical structure function given in (7).

This property, invariance to state permutations, is one of the extensions to the original definition of the dynamical structure function that will be developed in this paper:

1) Include noisy perturbations, which allows for the separation of perturbations inflicted by an external source, like an attacker or intrinsic noise, from controlled inputs.
2) Allow for non-strictly causal internal dynamics of systems, which allows for feed forward effects between manifest variables but also introduces potential well-posedness issues.
3) Define a transformation of the system that maintains invariance to state permutations and to decoupled state transformations on measured and hidden states.

Definition 2. *Consider a system of the form*

$$\begin{array}{rcl}\dot{x} &=& Ax + Bu + F\psi \\ y &=& Cx + Du + H\psi\end{array} \qquad (9)$$

*where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^l$, $u \in \mathbb{R}^m$, $\psi \in \mathbb{R}^r$, and A, B, C, D, F, and H are of appropriate dimension. The procedure for calculating the **dynamical structure function** is as follows:*

1) Let p be the rank of C, and assume without loss of generality that the outputs $y = [y_1' \quad y_2']'$, $y_1 \in \mathbb{R}^p$ and $y_2 \in \mathbb{R}^{(l-p)}$, are ordered so the first p rows of C are linearly independent, i.e.

$$C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

with $C_1 \in \mathbb{R}^{p\times n}$ being full row rank. The dynamical structure function of the system in (9) with respect to y_1 is then given by the $(l \times p)$, $(l \times m)$, and $(l \times r)$ real rational matrix functions, $(\hat{Q}(s), \hat{P}(s), \hat{R}(s))$, defined over the Laplace variable, $s \in \mathbb{C}$, and constructed by the following procedure:

2) Create the $(n \times n)$ state transformation:

$$T = \begin{bmatrix} C_1' & E_1 \end{bmatrix}', \qquad (10)$$

where $E_1 \in \mathbb{R}^{n\times(n-p)}$ is any basis of the null space of C_1, with

$$T^{-1} = \begin{bmatrix} R_1 & E_1 \end{bmatrix}, \qquad (11)$$

where $R_1 = C_1'(C_1C_1')^{-1}$.

3) Change basis such that $z = Tx$, yielding $\hat{A} = TAT^{-1}$, $\hat{B} = TB$, $\hat{C} = CT^{-1}$, $\hat{D} = D$, $\hat{F} = TF$, and $\hat{H} = H$, and partitioned commensurate with the block partitioning of T and T^{-1} to give

$$\begin{array}{rcl}\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} &=& \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \\ && \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix} u + \begin{bmatrix} \hat{F}_1 \\ \hat{F}_2 \end{bmatrix}\psi \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &=& \begin{bmatrix} I & 0 \\ \hat{C}_{21} & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \\ && \begin{bmatrix} \hat{D}_1 \\ \hat{D}_2 \end{bmatrix} u + \begin{bmatrix} \hat{H}_1 \\ \hat{H}_2 \end{bmatrix}\psi \end{array} \qquad (12)$$

Note that while it is easily seen that $C_1R_1 = I$ and $C_1E_1 = 0$, but the fact that $C_2E_1 = 0$ may demand some reflection. The reason this is true is because every row of C_2 is in the row space of C_1. If it were not so, then either the rank of C would be greater than p or C_1 would not be composed of p linearly independent rows. Being in the row space of C_1, each row in C_2 is thus also orthogonal to every vector in E_1, which spans the orthogonal complement of the row space of C_1.

4) Assume zero initial conditions, take Laplace transforms, and solve for Z_2, yielding

$$\begin{array}{rcl} sZ_1 &=& \left[\hat{A}_{11} + \hat{A}_{12}(sI - \hat{A}_{22})^{-1}\hat{A}_{21}\right] Z_1 \\ && + \left[\hat{B}_1 + \hat{A}_{12}(sI - \hat{A}_{22})^{-1}\hat{B}_2\right] U \\ && + \left[\hat{F}_1 + \hat{A}_{12}(sI - \hat{A}_{22})^{-1}\hat{F}_2\right] \Psi \\ \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} &=& \begin{bmatrix} I & 0 \\ \hat{C}_{21} & 0 \end{bmatrix}\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} + \\ && \begin{bmatrix} \hat{D}_1 \\ \hat{D}_2 \end{bmatrix} U + \begin{bmatrix} \hat{H}_1 \\ \hat{H}_2 \end{bmatrix}\Psi \end{array} \qquad (13)$$

where Z, U, Y, and Ψ denote the Laplace transforms of z, u, y, and ψ respectively.

5) For notational simplicity, define:

$$W(s) = \hat{A}_{11} + \hat{A}_{12}(sI - \hat{A}_{22})^{-1}\hat{A}_{21} \qquad (14)$$

$$V(s) = \hat{B}_1 + \hat{A}_{12}(sI - \hat{A}_{22})^{-1}\hat{B}_2 \qquad (15)$$

$$L(s) = \hat{F}_1 + \hat{A}_{12}(sI - \hat{A}_{22})^{-1}\hat{F}_2 \qquad (16)$$

and let $D_W(s) = \mathrm{diag}(W(s))$ be a diagonal matrix function composed of the diagonal entries of $W(s)$.

6) Define $Q(s) = (sI - D_W)^{-1}(W - D_W)$, $P(s) = (sI - D_W)^{-1}V$, and $R(s) = (sI - D_W)^{-1}L$ yielding

$$\begin{array}{rcl} Z_1 &=& Q(s)Z_1 + P(s)U + R(s)\Psi \\ \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} &=& \begin{bmatrix} I & 0 \\ \hat{C}_{21} & 0 \end{bmatrix}\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} + \\ && \begin{bmatrix} \hat{D}_1 \\ \hat{D}_2 \end{bmatrix} U + \begin{bmatrix} \hat{H}_1 \\ \hat{H}_2 \end{bmatrix}\Psi \end{array} \qquad (17)$$

7) Noting from (17) that $Z_1 = Y_1 - \hat{D}_1U - \hat{H}_1\Psi$, the dynamical structure function of (9) with respect to y_1 is then given by:

$$\hat{Q}(s) = \begin{bmatrix} Q(s) \\ C_{21} \end{bmatrix},$$

$$\hat{P}(s) = \begin{bmatrix} P(s) + (I - Q(s))\hat{D}_1 \\ \hat{D}_2 - \hat{C}_{21}\hat{D}_1 \end{bmatrix} \qquad (18)$$

$$\hat{R}(s) = \begin{bmatrix} R(s) + (I - Q(s))\hat{H}_1 \\ \hat{H}_2 - \hat{C}_{21}\hat{H}_1 \end{bmatrix}$$

which satisfies

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \hat{Q}(s)Y_1 + \hat{P}(s)U + \hat{R}(s)\Psi \qquad (19)$$

The non-strictly causal dynamical structure function can be defined from the subsystem structure using the above procedure if each subsystem is modeled in the form (9). First determine the dynamical structure functions $(\hat{Q}_i(s), \hat{P}_i(s), \hat{R}_i(s))$ for each subsystem S_i, then interconnect them using block diagram algebra to get the dynamical structure function of the overall system, $(\tilde{Q}(s), \tilde{P}(s), \tilde{R}(s))$.

Also, note that this new, extended definition of the dynamical structure function reduces to the previous definition in [1] for systems of the form

$$\begin{array}{rcl} \dot{x} & = & \hat{A}x + \hat{B}u \\ y & = & \left[\begin{array}{cc} I & 0 \end{array} \right] x. \end{array} \quad (20)$$

This is easily seen since these systems generate transformations of the form $T = T^{-1} = I$ in step (2) above and have $y = y_1$ (y_2 is empty), thereby leading to $\hat{Q} = Q$ and $\hat{P} = P$. In this way, this definition of the dynamical structure function is a natural generalization extending the earlier theory.

C. Properties of the Extended Dynamical Structure Function

In this section we detail invariance properties of the dynamical structure function as defined above.

Lemma 1. (Invariance to a Class of Block Diagonal Transformations) *Given a system* (A, B, C, D, F, H) *of the form (12) with dynamical structure function* $(\hat{Q}, \hat{P}, \hat{R})$*, then* $(\hat{Q}, \hat{P}, \hat{R})$ *is invariant to block diagonal state transformations; that is, the set of systems characterized by block diagonal state transformations,*

$$\begin{array}{rcl} \mathcal{S} & = & \{(MAM^{-1}, MB, CM^{-1}, D, MF, H) \\ & & \quad | \; M = \left[\begin{array}{cc} I_{p\times p} & 0 \\ 0 & M_{22} \end{array} \right] \}, \end{array}$$

with M_{22} *any invertible matrix of appropriate size, all share the same dynamical structure function,* $(\hat{Q}, \hat{P}, \hat{R})$.

Proof. Transforming the given system, $\bar{z} = Mz$, yields

$$\begin{array}{rcl} \left[\begin{array}{c} \dot{\bar{z}}_1 \\ \dot{\bar{z}}_2 \end{array} \right] & = & \left[\begin{array}{cc} A_{11} & A_{12}M_{22}^{-1} \\ M_{22}A_{21} & M_{22}A_{22}M_{22}^{-1} \end{array} \right] \left[\begin{array}{c} \bar{z}_1 \\ \bar{z}_2 \end{array} \right] + \\ & & \left[\begin{array}{c} B_1 \\ M_{22}B_2 \end{array} \right] u + \left[\begin{array}{c} F_1 \\ M_{22}F_2 \end{array} \right] \psi \\ \left[\begin{array}{c} y_1 \\ y_2 \end{array} \right] & = & \left[\begin{array}{cc} I & 0 \\ C_{21} & 0 \end{array} \right] \left[\begin{array}{c} \bar{z}_1 \\ \bar{z}_2 \end{array} \right] + \left[\begin{array}{c} D_1 \\ D_2 \end{array} \right] u + \\ & & \left[\begin{array}{c} H_1 \\ H_2 \end{array} \right] \Psi \end{array} \quad (21)$$

which leads to

$$\begin{array}{rcl} W(s) & = & A_{11} + A_{12}M_{22}^{-1}(sI - M_{22}A_{22}M_{22}^{-1})^{-1}M_{22}A_{21} \\ & = & A_{11} + A_{12}(sI - A_{22})^{-1}A_{21}, \end{array} \quad (22)$$

$$\begin{array}{rcl} V(s) & = & B_1 + A_{12}M_{22}^{-1}(sI - M_{22}A_{22}M_{22}^{-1})^{-1}M_{22}B_2 \\ & = & B_1 + A_{12}(sI - A_{22})^{-1}B_2. \end{array} \quad (23)$$

$$\begin{array}{rcl} L(s) & = & F_1 + A_{12}M_{22}^{-1}(sI - M_{22}A_{22}M_{22}^{-1})^{-1}M_{22}F_2 \\ & = & F_1 + A_{12}(SI - A_{22})^{-1}F_2. \end{array} \quad (24)$$

Since $W(s)$, $V(s)$, and $L(s)$ are invariant to M_{22}, $(\hat{Q}, \hat{P}, \hat{R})$ also remain unchanged with respect to M_{22}. □

Theorem 1. (Invariance to Basis of the Null Space) *Given a system* (A, B, C, D, F, H) *as in (9), consider two distinct bases of the null space of* C, $E \neq \bar{E}$, *with corresponding state transformations:*

$$T = \left[\begin{array}{c} C_1 \\ E' \end{array} \right], \quad \bar{T} = \left[\begin{array}{c} C_1 \\ \bar{E}' \end{array} \right],$$

as in (10), and each leading to its corresponding dynamical structure function, $(\hat{Q}, \hat{P}, \hat{R})$ *and* $(\bar{Q}, \bar{P}, \bar{R})$ *as in (18). Then* $(\hat{Q}, \hat{P}, \hat{R}) = (\bar{Q}, \bar{P}, \bar{R})$.

Proof. Let $z = Tx$ and $\bar{z} = \bar{T}x$. Then $\bar{z} = \bar{T}T^{-1}z$:

$$\bar{T}T^{-1} = \left[\begin{array}{c} C_1 \\ \bar{E}' \end{array} \right] \left[\begin{array}{cc} R_1 & E \end{array} \right] = \left[\begin{array}{cc} I & 0 \\ 0 & \bar{E}'E \end{array} \right] \quad (25)$$

where $R_1 = C_1'(C_1C_1')^{-1}$. The block diagonal structure of $\bar{T}T^{-1}$ then ensures, by Lemma 1, that the dynamical structure function produced for $\bar{z}$ is the same as that for z, i.e. $(\hat{Q}, \hat{P}, \hat{R}) = (\bar{Q}, \bar{P}, \bar{R})$. □

Theorem 1 ensures that the procedure detailed above for calculating the dynamical structure function is well defined for any system of the form (9).

Theorem 2. (Invariance to State Permutations) *Consider a system as in (9) with state matrices* (A, B, C, D, F, H) *and dynamical structure function* $(\hat{Q}, \hat{P}, \hat{R})$*. Then* $(\hat{Q}, \hat{P}, \hat{R})$ *is invariant to state permutations; that is, the set of systems characterized by state permutations,*

$$\begin{array}{rcl} \mathcal{S} & = & \{(MAM^{-1}, MB, CM^{-1}, D, MF, H) \\ & & \quad | \; M \text{ is a permutation matrix}\}, \end{array}$$

all share the same dynamical structure function, $(\hat{Q}, \hat{P}, \hat{R})$*, up to a permutation.*

Proof. The result follows from the fact that that the state transformation selected in Step 1 of constructing the dynamical structure function transforms each system in the set $\mathcal{S}$ to the same system for Step 2; the resulting dynamical structure function is thus the same, up to a permutation. To see this, consider the transformation T constructed for the unpermuted system, (A, B, C, D, F, H):

$$T = \left[\begin{array}{c} C_1 \\ E_1' \end{array} \right], \quad T^{-1} = \left[\begin{array}{cc} R_1 & E_1 \end{array} \right],$$

and compare with the transformation T_M constructed for any permuted system, $(MAM^{-1}, MB, CM^{-1}, D, MF, H)$:

$$\begin{array}{c} T_M = \left[\begin{array}{c} C_1 \\ E_1' \end{array} \right] M^{-1} = TM^{-1}, \\ T_M^{-1} = M \left[\begin{array}{cc} R_1 & E_1 \end{array} \right] = MT^{-1}. \end{array}$$

Applying each set of transformations to their respective systems yields the same transformed system, thus producing the same dynamical structure function:

$$\begin{array}{c} (T_M MAM^{-1}T_M^{-1}, T_M MB, CM^{-1}T_M^{-1}, D, T_M MF, H) \\ = (TAT^{-1}, TB, CT^{-1}, D, TF, H). \end{array}$$

□

Revisiting Example 2, we next see that the extended definition of the dynamical structure function given here preserves invariance of the dynamical structure function to state permutations, as stated in Theorem 2.

Example 3. *Consider the state space equation given in (6), using the new extended definition, the dynamical structure function is:*

$$Q(s) = \begin{bmatrix} 0 & 0 \\ \frac{-s-3}{(s+2)(2s+7)} & 0 \end{bmatrix},$$

$$P(s) = \begin{bmatrix} \frac{s+5}{s^2+4s+1} & 0 & \frac{s+2}{s^2+4s+1} \\ \frac{1}{(s+2)(2s+7)} & \frac{1}{s+2} & \frac{-s-2}{2s+7} \end{bmatrix} \tag{26}$$

Using a state permutation to renumber the first and third states, as in Example 2, and calculate the dynamical structure function as

$$Q(s) = \begin{bmatrix} 0 & 0 \\ \frac{-s-3}{(s+2)(2s+7)} & 0 \end{bmatrix},$$

$$P(s) = \begin{bmatrix} \frac{s+5}{s^2+4s+1} & 0 & \frac{s+2}{s^2+4s+1} \\ \frac{1}{(s+2)(2s+7)} & \frac{1}{s+2} & \frac{-s-2}{2s+7} \end{bmatrix} \tag{27}$$

which is consistent with the dynamical structure function without permutations in (26).

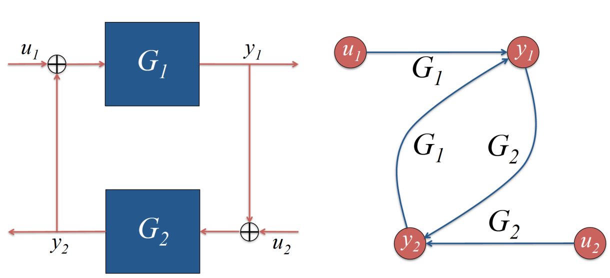
IV. Semantics of Partial Structure Representations

The differences between the interconnections of subsystems and dynamical structure functions will now be demonstrated by comparing the set of all state realizations consistent with one to the set of all state realizations consistent with the other.

Example 4. *Consider the feedback interconnection of two subsystems, G_1 and G_2, as shown in Figure 1a. Interconnecting these subsystems yields the closed-loop system, G_{CL}, given by:*

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} \frac{G_1}{1-G_1G_2} & \frac{G_1G_2}{1-G_1G_2} \\ \frac{G_2G_1}{1-G_1G_2} & \frac{G_2}{1-G_1G_2} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \tag{28}$$

Figure 1a illustrates the subsystem structure *of G_{CL}, or the interconnection pattern of subsystems generating the dynamics seen externally as G_{CL}.*



(a) *Subsystem structure* is a graph with nodes labeled as systems and edges labeled as signals, typically drawn as a block diagram.

(b) *Signal structure* is a graph with nodes labeled as signals and edges labeled as systems, typically drawn as a signal-flow graph.

Fig. 1: Two structural views of the same system.

Figure 1b illustrates a different decomposition of G_{CL}, characterized by the following relationships among the manifest signals:

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 0 & G_1 \\ G_2 & 0 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} + \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \tag{29}$$

Note that this characterization defines the graphical structure shown in Figure 1b and is consistent with G_{CL}, since Equation (29) implies:

$$\begin{aligned} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} &= \left(I - \begin{bmatrix} 0 & G_1 \\ G_2 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{G_1}{1-G_1G_2} & \frac{G_1G_2}{1-G_1G_2} \\ \frac{G_2G_1}{1-G_1G_2} & \frac{G_2}{1-G_1G_2} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}. \end{aligned} \tag{30}$$

While the structural information in Figures 1a and 1b may appear redundant, the subsystem and signal structures represent very different sets of possible realizations of G_{CL}.

To make the set comparison concrete, further consider each system in the feedback interconnection in Figure 1a to be given by

$$G_1 = \frac{s^2+3s+3}{(s+1)(s+2)} \text{ and } G_2 = \frac{1}{(s+3)(s+4)} \tag{31}$$

Knowing the subsystem structure of G_{CL}, as depicted in Figure 1a, eliminates all associated realizations that are not fourth order. Moreover, this set of fourth order realizations allowed by the subsystem structure can be completely characterized by:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} A_1 & B_1C_2 \\ B_2C_1 & A_2 + B_2D_1C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ B_2D_1 & B_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= \begin{bmatrix} C_1 & D_1C_2 \\ 0 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \end{aligned} \tag{32}$$

where $x_1, x_2 \in \mathbb{R}^2$, and (A_i, B_i, C_i, D_i), $i = 1, 2$ are any minimal realization of G_1 and G_2, respectively.

The dynamical structure function associated with the subsystem structure in (31) is

$$Q(s) = \begin{bmatrix} 0 & \frac{s^2+3s+3}{(s+1)(s+2)} \\ \frac{1}{(s+3)(s+4)} & 0 \end{bmatrix}$$

$$P(s) = \begin{bmatrix} \frac{s^2+3s+3}{(s+1)(s+2)} & 0 \\ 0 & \frac{1}{(s+3)(s+4)} \end{bmatrix} \tag{33}$$

if we take into account the subsystem structure of the system a priori.

If the subsystem structure was unknown, then the dynamical structure function derived from (32) is given by:

$$Q(s) = \begin{bmatrix} 0 & \frac{(s+4)(-2s-3)}{(s+2)(s^2+5s+3)} \\ \frac{1}{(s+3)(s+4)} & 0 \end{bmatrix}$$

$$P(s) = \begin{bmatrix} \frac{(s+4)(s^2+3s+3)}{(s+2)(s^2+5s+3)} & \frac{(s+3)(-2s-6)}{(s+2)(s^2+5s+3)} \\ 0 & \frac{s^2+7s+13}{(s+3)(s+4)} \end{bmatrix} \tag{34}$$

The knowledge of the subsystem structure allows $Q(s)$ to be non-strictly causal, whereas when the subsystem structure *is not known a priori, $Q(s)$ is always strictly causal. Next we compare and contrast the set of realizations consistent with the subsystem structure in (31) and the set of realizations consistent with the signal structure in (34).*

1) *A minimal (stable) realization consistent with the subsystem structure in (31) and the dynamical structure function in (34) is*

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -3 & 1 \\ 1 & 0 & 1 & -4 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} u \end{aligned} \tag{35}$$

This shows that the sets of realizations consistent with subsystem and signal structures overlap.

2) *Consider now a transformation of the form*

$$\begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix} \text{ where } M = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

on the state space given in (35), after permutation to ensure $C = \begin{bmatrix} I & 0 \end{bmatrix}$. From Lemma 1 we know that the dynamical structure function doesn't change.
The state space representation then becomes, after permutation,

$$\begin{aligned} \dot{\hat{x}} &= \begin{bmatrix} -1 & 1 & -2 & 0 \\ 1 & 0 & 1 & -2 \\ 0 & -1 & -3 & 1 \\ 2 & 4 & 1 & -6 \end{bmatrix} \hat{x} + \begin{bmatrix} 0 & 0 \\ 2 & 1 \\ 0 & 0 \\ 3 & 2 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \hat{x} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} u \end{aligned} \tag{36}$$

Note that C and B do not have the form given in (32) and no state permutations will put them in the correct form. This demonstrates that there exist realizations consistent with the dynamical structure function (34) but which are not consistent with the subsystem structure in (31), even up to a permutation.

3) *Finally, it can be shown that the set of state transformations that preserve the structure in (31) are equivalent to the set of all state space transformations that are block diagonal (i.e. transform the state of each subsystem independently). Consider now a transformation of the form*

$$\begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \text{ where } M = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

on the state space given in (35). This transformation maintains the subsystem structure; however, the corresponding dynamical structure function becomes

$$\begin{aligned} Q(s) &= \begin{bmatrix} 0 & \frac{(s+1)(-10s-33)}{5s^3+32s^2+56s+27} \\ \frac{1}{(s+3)(s+4)} & 0 \end{bmatrix} \\ P(s) &= \begin{bmatrix} \frac{5s^3+32s^2+66s+51}{5s^3+32s^2+56s+27} & \frac{10s^2+48s+40}{5s^3+32s^2+56s+27} \\ 0 & \frac{s^2+7s+13}{(s+3)(s+4)} \end{bmatrix} \end{aligned} \tag{37}$$

It follows that for this example, the set of realizations consistent with the subsystem structure in (31) is different from, but overlaps with, the set of realizations consistent with the signal structure, characterized by the dynamical structure function in (34).

V. Conclusion

In this paper we extended the definition of the dynamical structure function to allow for non-strictly causal representations and maintain invariance to state permutations. We introduced a method for characterizing the semantics of partial structure representations of network systems, and illustrated the method by demonstrating that a system's signal and subsystem structures are quite distinct [16].

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