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Dynamic Coalition Formation in Markets with Linear Demand

by

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Abstract

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This thesis presents a gradient play differential game in a producer market with linear demand as a model of behavior in a competitive environment. This model is then extended to allow for cooperation. Definitions of coalitions and natural coalitions are provided, and previous work dealing with static coalition structure is presented. Requirements for acceptable system dynamics are given, and a specific set of dynamics presented. These dynamics are shown to meet the specified requirements in a number of cases through proofs of stability of equilibria as well as guarantees that the dynamics result in natural coalitions. Numerical examples are provided to support the claim that a reasonable equilibrium exists and that its basin of attraction covers reasonable initial conditions. The work concludes with a summary as well as possible paths for continued research.

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Abbreviations

NCE	**N**on-**C**ooperative **E**quilibrium
PCE	**P**erfecftly **C**ooperative **E**quilibrium
CE	**C**ooperative **E**quilibrium

Chapter 1

Introduction

Cooperation in competitive environments appears in a variety of fields, from business to biology. Traditionally, questions relating to the cooperation of rational agents in these environments belong to the purview of cooperative game theory. In that context, the system is as a coalitional game with transferable payoffs, and a characteristic function determines the total utility produced by that coalition. The Shapley value indicates the value each participant contributes to the cooperative group (coalition) and dictates distribution [1].

However, this representation fails to capture the effects of one coalition's choices on the payoffs of other coalitions [2]. The characteristic function of cooperative games maps any subset of the agents to some payoff for that coalition but ignores the coalitional structure of the other firms not included in the coalition in question. Though variations which attempt to take this external structure into account exist, a systems and control approach to this problem offers distinct advantages for representing and reasoning about cooperation in competitive environments.

Certain representations of agents' behaviors in markets, especially gradient play differential games, lend themselves to control and systems analysis. By modeling firms' decision strategies with differential equations and using a vector of the firms' current strategies as a state vector, the game can be represented as a mathematical system. This allows for the use of analytical methods to investigate properties of the system, such as stability. Although much of the vocabulary in this work will be taken from economics and game theory literature, the market dynamics abstract to other types of systems. Ultimately, the problem here is algorithmic, not economic.

Some work has already be done in this area (see [3], [4] and [5]). However, these have all considered cases in which cooperative structure is defined *a priori*, which can lead to situations in which cooperation is in fact not beneficial for all involved. This work presents a set of dynamics which allows cooperative structure to change over time and guarantees that all participants are better off at equilibrium. Designing dynamics which act this way yields two important benefits. First, they can be used to model systems where cooperation already occurs, such as team dynamics in sports, firm formation in industrial organization or agent coordination for autonomous systems. Second, these principles can be used to induce cooperation in situations where cooperation would be beneficial but does not occur naturally, such as distributed computation.

1.1 Market Model

A gradient play differential game can represent the behavior of rational agents in competitive environment. The game consist of of a set N of n firms who participate in a Bertrand producer market. The market includes a price vector $p \in \mathbb{R}^n$, a demand function $q(p) : \mathbb{R}^n \mapsto \mathbb{R}^n$ and profit function $\pi(p) : \mathbb{R}^n \mapsto \mathbb{R}^n$. The price, demand and profits associated with a specific firm $i \in N$ are denoted with a subscript: $p_i \in \mathbb{R}$, $q_i(p) : \mathbb{R}^n \mapsto \mathbb{R}$ and $\pi_i(p) : \mathbb{R}^n \mapsto \mathbb{R}$. Each firm sells an ordinary, differentiable good at a price p_i which the firm determines. Cross-price demand may exist in the market, meaning that the demand for each good is a function of all prices. All firms are assumed to produce at zero cost, so profits are given by the product of price and demand.

$$\pi_i = p_i q_i(p) \tag{1.1}$$

While individual profits tell us how each firm fares, the sum of all profits measures producer performance as a whole.

Definition 1.1. The sum over all profits in the market is called *Producer Surplus*, w:

$$w(p) = \sum_{i \in N} \pi_i(p). \tag{1.2}$$

Definition 1.2. The *Maximum Producer Surplus* or *Cartel Outcome* w^* is the maximum producer surplus the market can achieve over all possible price combinations when such a maximum exists, otherwise the maximum producer surplus is understood to be infinity:

$$w^* = \max_p w(p). \tag{1.3}$$

Definition 1.3. An *Optimal Pricing Policy* p^* is a set of prices which result in the market producing maximum producer surplus:

$$p^* = \underset{p}{\operatorname{argmax}}\ w(p). \tag{1.4}$$

Note that depending on the demand function, a market may have multiple optimal pricing policies, all of which produce the same producer surplus.

1.1.1 Demand Function

This work considers linear demand functions, which have the form

$$q(p) = Ap + b, \quad A \in \mathbb{R}^{n \times n}, b \in \mathbb{R}^n. \tag{1.5}$$

Note that entry a_{ij} represents the sensitivity of demand for good i with respect to changes in price of good j. This sensitivity of demand will be positive for substitute goods and negative for complementary goods, and the relationship between goods as complements or substitutes is symmetric. Because the goods sold are assumed to be normal, an increase in a good's price always decreases the demand for that good, so diagonal entries must be negative. Additionally, a firm's own price affects its demand more than the combined influence of all other goods, and affects its own demand more than it affects all other goods' demands combined. These restrictions give rise to the following definition:

Definition 1.4. A linear demand function $q(p) = Ap + b$ is said to be *admissible* if the following conditions are met:

1. A and A^T are diagonally dominant.
2. A is sign symmetric.
3. $\forall i \in N$, $a_{ii} < 0$.
4. $\forall i \in N$, $b_i > 0$.

The following definition relating to the properties of demand functions will also be useful throughout this work:

Definition 1.5. A linear demand function $q(p) = Ap + b$ is said to have non-zero cross-price elasticity of demand if A is not diagonal.

The requirement of diagonal dominance on both A and A^T ensures that A and $(A + A^T)$ are invertible. This in turn ensures the existence of exactly one optimal pricing policy, which can be found analytically. For $p \in \mathbb{R}^n$, let $diag(p) = D \in R^{n\times n}$ denote a diagonal matrix where $d_{ii} = p_i$. Let $\mathbf{1}$ indicate an appropriately sized vector of 1's. Producer surplus may be written as

$$w = \mathbf{1}^T\ \Pi(p) = \mathbf{1}^T diag(p)q(p) = p^T(Ap + b). \tag{1.6}$$

The maximum is found by taking the derivative with respect to p and setting that expression equal to 0:

$$\frac{\partial w}{\partial p} = (A + A^T)p + b = 0. \tag{1.7}$$

Because $(A+A^T)$ is invertible, there is exactly one optimal pricing policy given by $p^* = (A + A^T)^{-1}b$, and an individual firm's price expressed in terms of the prices of the other firms is

$$p_i^* = \frac{\sum_{j\in N/i}(a_{ij} + a_{ji})p_j^* + b_i}{-2a_{ii}}. \tag{1.8}$$

Therefore, any market characterized by an admissible linear demand function has exactly one optimal pricing policy, and any other pricing policy leaves unrealized profits in the system from the point of view of the producers. Analysis of consumer surplus is ignored.

1.1.2 Dynamics

In gradient play differential games, each firm's strategy is defined by the gradient of its payoff function [6]. In the current context, a firm's strategy is the price set for the corresponding good, and its strategy set is restricted to positive real numbers. The following differential equation represents a firm's pricing decisions:

$$\dot{p}_i = \frac{\partial U_i}{\partial p_i}. \tag{1.9}$$

where $U_i \in \mathbb{R}$ is a firm's utility, an abstract measure of personal benefit. Here and throughout the rest of this work, '$\cdot$' indicates a time derivative of a variable. Two fundamental definitions of individual firms' utility, one based on each firm's own profits and the other based on producer surplus, and resulting dynamics are presented to illustrate the problem this work seeks to address.

1.1.2.1 No Cooperation

In the absence of cooperation, each firm seeks to maximize its own profit with no regard whatsoever to the other firms' profits. Letting $U_i = \pi_i$ and applying (1.9) results in

$$\dot{p}_i = \frac{\partial \pi_i}{\partial p_i} = q_i(p) + p_i \frac{\partial q_i}{\partial p_i}. \tag{1.10}$$

Dynamics for the entire system are then:

$$\dot{p} = (A + \hat{A})p + b. \tag{1.11}$$

where $\hat{a}_{ii} = a_{ii}$ and $\hat{a}_{ij} = 0$. The stability of the system described by (1.11) is deduced by noting that $(A + \hat{A})$ is diagonally dominant and the diagonal entries are negative, which implies the real parts of the eigenvalues of the matrix are also negative. Also, since $(A + \hat{A})$ is invertible, there exists a unique equilibrium for the system. At this non-cooperative equilibrium (NCE), each firm's price is

$$p_i^{nce} = \frac{\sum_{j \in N/i} a_{ij} p_j^{nce} + b_i}{-2a_{ii}}. \tag{1.12}$$

We now compare the optimal pricing strategy with the prices in market at non-cooperative equilibrium. For an admissible linear demand function with non-zero cross-price elasticity of demand, $p_i^{nce} \neq p^*$, which implies that $w^{nce} < w^*$. However, if there cross-price elasticity is zero, then the $p_i^{nce} = p^*$ and $w^{nce} = w^*$. This essentially means that there is no benefit to cooperation if there is zero cross-price elasticity of demand, and therefore, in subsequent analysis we will generally consider markets where it is non-zero.

1.1.2.2 Perfect Cooperation

In the perfectly cooperative case, all producers collude perfectly to achieve maximum producer surplus, so that

$$U_i = w = \sum_{i \in N} \pi_i \tag{1.13}$$

which results in the following dynamics:

$$\begin{aligned} \dot{p}_i &= \frac{\partial w}{\partial p_i} = \sum_{j=1}^{n} \frac{\partial \pi_j}{\partial p_i} \\ \dot{p} &= (A + A^T)p. \end{aligned} \tag{1.14}$$

Similar to the non-cooperative case, the existence of a single stable equilibrium given an admissible demand function becomes apparent upon recognition that $(A + A^T)$ is also diagonally dominant and the diagonal entries are negative. This equilibrium will be referred to as the perfectly cooperative equilibrium or PCE.

Note that these dynamics mirror (1.7), and at the PCE, $p^{pce} = p^*$. Importantly, this implies that producer surplus at the PCE is always at least as great as it is at the NCE. This indicates that cooperation allows the market to realize an elevated amount of producer surplus compared to that obtainable without cooperation. However, although perfectly cooperative dynamics result in maximum producer surplus, there is no guarantee that all firms are better off individually.

For example, consider the two-firm market characterized by the following admissible demand function:

$$q(p) = \begin{bmatrix} -3 & -2 \\ -1 & -3 \end{bmatrix} p + \begin{bmatrix} 6 \\ 6 \end{bmatrix} \tag{1.15}$$

The prices, profits and producer surplus of the system under non-cooperative and perfectly cooperative equilibrium are shown in the table below.

Dynamics	$p_1(\$)$	$p_2(\$)$	$\pi_1(\$)$	$\pi_2(\$)$	$w(\$)$
Non-Cooperative	0.71	0.88	1.49	2.34	3.83
Perfectly Cooperative	0.67	0.67	1.78	2.22	4.00

Table 1.1: Prices, profits and producer surplus in the market given in (1.15) at equilibrium under different dynamics.

Notice that although producer surplus is higher with perfectly cooperative equilibrium, the second firm is actually worse off than it would be under non-cooperative dynamics. In cases such as that given above, we can assume that the firm either does value the total producer surplus for some unspecified reason or that there exists some hidden redistribution mechanism which allocates producer surplus in such a way that everyone is better off. Because this work investigates cooperation in competitive environments, we wish to consider the case where firms are motivated strictly by the money they receive, which precludes wholly altruistic behavior. This work considers side payments between firms as a means of redistributing profits in such a way that firms are incentivized to cooperate without altruistic motivation.

1.2 Coalitions and Side Payments

Coalitions and side payments provide a framework for explicitly representing cooperation and redistribution. A coalition is a set of firms whose utility takes into account the profits of other firms in the coalition. More formally,

Definition 1.6. A *coalition* C is a set of firms such that

$$\begin{aligned} &\forall i \in C, U_i = \sum_{j \in C} \beta_{ij}\pi_j \\ &0 \leq \beta_{ij} \leq 1, \quad \beta_{ij} = 0 \implies \beta_{ji} \neq 0 \end{aligned} \tag{1.16}$$

This general definition allows us to consider perfectly cooperative dynamics as the behavior of the system when everyone participates in a single, all encompassing coalition (referred to as the grand coalition) and the cooperation coefficients β are all equal to 1. However, this general definition also allows purely altruistically motivated cooperation. Two definitions are useful to specify the type of cooperation that is of interest here.

Definition 1.7. A firm's *revenue* v_i is the total amount of money a firm has after paying and receiving side payments.

Definition 1.8. A *natural coalition* C_N is a set of firms such that

$$\begin{gathered}\forall i \in C_N,\ \ \pi_i^{nce} \leq v_i = \pi_i + \sum_{j \in C \setminus i} \beta_{ij}\pi_j - \sum_{j \in C \setminus i} \beta_{ji}\pi_i, \\ \sum_{j \in C} \beta_{ji} = 1, \quad 0 \leq \beta_{ij} \leq 1, \quad \beta_{ij} = 0 \implies \beta_{ji} \neq 0.\end{gathered} \tag{1.17}$$

where π_i^{nce} are firm i's profits at the NCE. The side payment from firm j to firm i is the value $\beta_{ij}\pi_j$. Side payments cause firms' personal interests to align, partially at least, with the interests of other firms as well for selfish rather than altruistic reasons. Thus acting selfishly, they may also benefit the other members of the coalition. This can lead to situations where all members of the coalition are better off than they would be acting independently.

1.3 Conclusion

The goal of this work is to define an algorithm representing rules for side payments between firm in the market which produces natural coalitions. These rules will be represented as differential equations and included as part of the system dynamics for the market. Notions of equilibrium and stability will be used to prove that the algorithm converges to correct values for prices and side payments, and a rigorous definition of 'correct' will be provided in Chapter 3.

The following chapters expand on dynamic system representations of cooperation in competitive environments. Chapter 2 reviews full and partial participation in fixed coalitions, providing conditions for stability and exposing the need for a representation of dynamic coalition formation. Chapter 3 presents dynamic coalition formation. It enumerates criteria for admissible side payment dynamics and proposes a set of dynamics which meet these criteria. Numerical examples are given to compare these dynamics to both non-cooperative and perfectly cooperative dynamics. Chapter 4 concludes the work by summarizing key points and presenting paths for further research.

Chapter 2

Fixed Coalition Structures

The Information and Decision Algorithms Laboratories at BYU has built up interesting results regarding the stability [4], [3] and value of cooperation [5] with coalition structure given *a priori*. In [4], it is shown that given certain properties of the demand function, proving the stability of the Grand Coalition, $G = N$ (when all firms participate in the same coalition together), suffices to show that all possible coalition structures have a single, stable equilibrium. In [3], these results are extended to coalition structures in which firms may partially participate. This chapter reviews this work in light of our goal to dynamically create natural coalitions.

2.1 Fixed Full Cooperation

Fixed full cooperation refers to the situation in which, before the game begins, each firm is assigned to participate in exactly one coalition. Consistent with the market model given earlier, profits are equal to the product of price and demand. However, the explicit inclusion of coalitions in the model requires a slight augmentation. For a coalition F, the objective function is

$$U_F = \sum_{i \in F} \pi_i(p) \tag{2.1}$$

Each firm seeks to to maximize the total profits of its corresponding coalition. For a firm $i \in F$, this results in the following price dynamics:

$$\dot{p}_i = \frac{\partial U_F}{\partial p_i} = \frac{\partial\left(\sum_{j \in F} \pi_j\right)}{\partial p_i} = \sum_{j \in F} \frac{\partial \pi_j}{\partial p_i}. \tag{2.2}$$

Dynamics for the entire system become

$$\dot{p} = [D_{\mathscr{F}}(J_q^T p))]p + q(p) \tag{2.3}$$

where J_q is the Jacobian of the demand function $q(p)$ and $D_{\mathscr{F}}(J_q^T p))$ is defined as: $d_{ij} = a_{ij}$ if i and j are in the same coalition, and $d_{ij} = 0$ otherwise. Fixed full cooperation can be thought of as a generalization of perfect cooperation. Each firm's goal is to maximize the profits of the coalition to which it belongs, and when the grand coalition defines the cooperative structure of the market, the system becomes identical to that described by the perfectly cooperative dynamics.

The central theorem of [4] addresses the question of which cooperative structures will result in a stable equilibrium. Let $\sigma(A)$ denote the spectrum of A.

Theorem 2.1. *Consider a market with a set of firms N and the objective function for the Grand Coalition G in (2.1). The system described in (2.3) has a unique stable equilibrium for all $\mathscr{F} \in \Delta$, where Δ is the set of all possible partitions of N, if for some positive real number ϵ,*

$$max \quad \sigma(H(p)) \leq -\epsilon, \quad \forall p \in \mathbb{R}^n$$

where $H(p)$ is the Hessian matrix of the objective function U_G.

Note for an admissible linear demand curve $q(p) = Ax+p$, $H(p) = (A+A^T)$, which satisfies the condition stated in Theorem 2.1. Therefore, all coalition structures result in a unique, stable equilibrium for any market characterized by an admissible linear demand curve.

Explicit distribution of a firm's revenues among participating firms is ignored, and there is no guarantee that the given cooperative structure results in natural coalitions. The failure to represent side payments is remedied to a certain extent in representations of partial cooperation.

2.2 Fixed Partial Cooperation

To represent the possibility of a firm participating in one or more coalitions to different degrees (partial cooperation), the two matrices, Φ and D are introduced. For a market with a firm set $N = \{1, \cdots, n\}$, Φ is a participation matrix, where $0 \leq \Phi_{ij} \leq 1$ indicates the degree to which firm i participates with firm j. Participation is assumed to be symmetrical, so that $\Phi_{ij} = \Phi_{ji}$, and each firm cooperates fully with itself, $\Phi_{ii} = 1$. The distribution matrix D is a diagonal, positive definite matrix. Note that the product $\hat{D} = D\Phi$ must be a stochastic matrix in order to preserve profits in the system. Entry $\hat{d}_{ij}$ represents the percentage of firm j's profit firm i receives, and $\hat{x}_{ij}\pi_i$ represents the side payment from firm j to firm i.

Let $A \circ B$ denote the Hadamard product or pointwise multiplication of two matrices of the same size. The resulting utility function and price dynamics for the system are

$$\begin{aligned} U &= D\Phi\Pi \\ \dot{p} &= \frac{\partial U}{\partial p} = D((\Phi \circ J_q^T(p))p + q(p)). \end{aligned} \tag{2.4}$$

A central theorem in [3] describes sufficient conditions which guarantee the existence of a unique, stable equilibrium for the system.

Theorem 2.2. *For the game described in (2.4) with firm set N and any positive definite, diagonal D, the system will have a unique, stable equilibrium for all positive semi definite Φ if there exists a positive real number ϵ such that*

$$max \quad \sigma(R_G(p)) \leq -\epsilon, \quad \forall x \in \mathbb{R}^n,$$

where $R_G(p)$ is the Jacobian of the system dynamics corresponding to the Φ associated with the Grand Coalition.

This result closely mirrors that found for full cooperation, and in fact fixed partial cooperative dynamics can also model full cooperation by restricting the values in Φ so that $\Phi_{ij} = 1$ if i and j are in the same coalition and $\Phi_{ij} = 0$ otherwise. The D matrix could be any diagonal, positive definite matrix, though should we desire $\hat{D}$ to represent actual side payments rather than a general notion of utility derived from another firm's profits, D is restricted so that the product $\hat{D}$ is a stochastic matrix.

2.3 Distribution

This brings up an important point. For some values of Φ, there exist multiple positive definite D matrices which satisfy the condition that the product $D\Phi$ be a stochastic matrix. Furthermore, different values for D lead to different individual revenues. For example, consider a market described by the demand function

$$q(p) = \begin{bmatrix} -4 & 2 \\ 1 & -3 \end{bmatrix} p + \begin{bmatrix} 8 \\ 9 \end{bmatrix} \tag{2.5}$$

and a participation matrix

$$\Phi = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \tag{2.6}$$

Both the following matrices are valid D matrices.

$$D_1 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \qquad D_2 = \begin{bmatrix} 0.58 & 0 \\ 0 & 0.42 \end{bmatrix} \tag{2.7}$$

Firms' profits and producer surplus at equilibrium for the different D matrices as well as for the non-cooperative equilibrium are shown in the table below.

Cooperative Structure	p_1(\$)	p_2(\$)	π_1(\$)	π_2(\$)	w(\$)
Non-Cooperative	1.38	1.28	39.84	28.01	67.85
D_1	1.64	1.48	34.715	34.715	69.43
D_2	1.64	1.48	40.27	29.16	69.43

Table 2.1: Prices, profits and producer surplus in the market given in (2.5) and (2.6) at equilibrium under different for different cooperative structures.

With distribution dictated by D_1, the first firm is worse off than it would be acting independently, while under D_2 the first firm is better off. However, given D_1, there is no way for the first firm to defect and act independently or perhaps even negotiate for a more favorable distribution such as in D_2. Although representation of cooperation with fixed partial participation in coalitions does explicitly lay out side payments, there is no guarantee that the *a priori* structure produces natural coalitions.

2.4 Conclusion

Market models with fixed cooperative structures have been well studied, and conditions which lead to a single, stable equilibrium are well defined. The system with only full cooperation in coalitions equates individual firms' utilities to the total profits of the coalition, ignoring how redistribution may occur if it occurs at all. Partial cooperation can represent side payments explicitly, but predefined coalition structures have no guarantee that the coalitions will be natural at equilibrium. This highlights the necessity of dynamic coalition formation which will guarantee natural coalitions.

Chapter 3

Dynamic Coalition Formation

Previous representations have examined cooperation in systems with fixed coalition structures. We now turn to consider the problem of modeling cooperation when the topology of cooperation changes over time. As explained in earlier chapters, fixed coalition structures may result in unnatural coalitions, meaning that a firm's participation in a cooperative coalition actually make it worse off than acting independently would have. These firms stay in coalitions not because it is beneficial but because they are compelled to by the constraints of the game. The rules designed here to govern dynamic coalition formation aim to ensure that cooperation is never coerced and exists only to the extent that it is selfishly beneficial.

This chapter presents specific rules for pricing and side payments in a two firm market which fulfill two specific criteria. First, all decisions are selfish, that is, the rules reflect the firms' desire to maximize their own revenues. Second, both firms must be at least as well off as they would be not cooperating. This is analogous with saying that any collusion must result in a natural coalition. Ideally, these dynamics would result in an optimal pricing strategy, though the goal here is to maximize individual revenues, not producer surplus.

3.1 An Updated Model

Models of static coalition structures employed matrices indicating agents' participation and degree of participation in the different coalitions, and side payments were represented as a percent of total profits given to another firm. For dynamic coalition formation, we decompose the price variables into two new variables to facilitate reasoning about effects of cooperation relative to no cooperation. Let price now be given by $p_i = \bar{p}_i + \hat{p}_i$. The first set of new price variables, $\bar{p}$, represent the price that the firm would set if it acted completely independently while the second, $\hat{p}$ represent a deviation from the non-cooperative price due to cooperation. The expression for an individual's profits under the new basis becomes

$$\begin{aligned} \pi_i &= p_i q_i(p) = (\bar{p}_i + \hat{p}_i)(A_{ii}(\bar{p}_i + \hat{p}_i) + A_{ij}(\bar{p}_j + \hat{p}_j) + b_i) \\ &= (\bar{p}_i + \hat{p}_i)(A_{ii}(\bar{p}_i + \hat{p}_i) + A_{ij}\bar{p}_j + b_i) + (\bar{p}_i + \hat{p}_i)A_{ij}\hat{p}_j. \end{aligned} \tag{3.1}$$

The above decomposition shows that the deviation of firm j from its non-cooperative strategy $\hat{p}_j$, results in a change in demand for the good produced by firm i equal to $A_{ij}\hat{p}_j$. This in turn changes profits by an amount equal to $p_i A_{ij}\hat{p}_j$. A side payment from a firm i to a firm j is represented as a percentage β of firm i's profits which occur because of the cooperative shift $\hat{p}_j$ of firm j:

$$\beta_i p_i A_{ij}\hat{p}_j, \quad 0 \leq \beta \leq 1. \tag{3.2}$$

Under this formulation, a firm's total revenue becomes the sum of profits, side payments given, and side payments received, and utility is strictly equal to revenue:

$$U_i = v_i = \pi_i - \beta_i A_{ij} p_i \hat{p}_j + \beta_j A_{ji} p_j \hat{p}_i. \tag{3.3}$$

We now describe a formulation for price dynamics which allows us to reason about the system with respect to NCE. Note that, holding $\bar{p}_i$ constant, $\frac{\partial \hat{p}_i}{\partial p_i} = 1$. As before, we take the partial derivative of utility with respect to price:

$$\begin{aligned}\dot{p}_i = \frac{\partial U_i}{\partial p_i} &= 2A_{ii}p_i + A_{ij}p_j + b_i - \beta_i A_{ij}\hat{p}_j + \beta_j A_{ji}p_j \\ &= 2A_{ii}\bar{p}_i + A_{ij}\bar{p}_j + b_i + 2A_{ii}\hat{p}_i + A_{ij}\hat{p}_j - \beta_i A_{ij}\hat{p}_j + \beta_j A_{ji}p_j.\end{aligned} \tag{3.4}$$

Let

$$\dot{\bar{p}}_i = 2A_{ii}\bar{p}_i + A_{ij}\bar{p}_j + b_i. \tag{3.5}$$

Since $p_i = \bar{p}_i + \hat{p}_i$, $\dot{p}_i = \dot{\bar{p}}_i + \dot{\hat{p}}_i$, and

$$\begin{aligned}\dot{\hat{p}}_i &= 2A_{ii}\hat{p}_i + A_{ij}\hat{p}_j - \beta_i A_{ij}\hat{p}_j + \beta_j A_{ji}p_j \\ &= 2A_{ii}\hat{p}_i + (1-\beta_i)A_{ij}\hat{p}_j + \beta_j A_{ji}p_j.\end{aligned} \tag{3.6}$$

The definition of $\dot{\bar{p}}_i$ in (3.5) parallels the definition of non-cooperative price dynamics given in (1.11). Because the $\bar{p}$ variables are completely decoupled from the other variables in the system, $\bar{p}_i$ in this cooperative system behaves exactly as p_i does in the non-cooperative system. Discussion in Chapter 1 shows that, given an admissible linear demand function, there is a single, stable set of prices to which the system converges, the NCE. Since here, in the cooperative system, $\bar{p}_i$ evolves in the exact same way, it too will always converge to the NCE given an admissible linear demand function, and at the cooperative equilibrium (CE), $\bar{p}^{ce} = p^{nce}$.

Also, because $\dot{\bar{p}}$ is independent of β and $\hat{p}$ and both $\dot{\beta}$ and $\dot{\hat{p}}$ are dependent on $\bar{p}$, we know that β and $\hat{p}$ will not reach their cooperative equilibrium values until $\bar{p}$ has reached its equilibrium value of p^{nce}. Therefore we consider $\bar{p}$ to be a constant when considering the system at or very near the cooperative equilibrium. This allows us to focus our analysis on the behavior of $\hat{p}_i$ and β_i. This assumption also implies that $2A_{ii}\bar{p}_i + A_{ij}\bar{p}_j + b_i = 0$, since at the NCE, the time derivative of $\bar{p}_i = 0$ (Equation (3.5)).

Now we can reason about a firm's total payoff relative to its profits at the NCE. As $\hat{p}_i$ is conceptually a deviation from the NCE, let $\hat{\pi}_i$ represent gains or losses in profits relative to $\pi_i^{nce} = \bar{p}_i(A_{ii}\bar{p}_i + A_{ij}\bar{p}_j + b_i)$:

$$\begin{aligned}
\pi_i &= \bar{p}_i(A_{ii}\bar{p}_i + A_{ij}\bar{p}_j + b_i) + \bar{p}_i A_{ij}\hat{p}_j + \hat{p}_i(A_{ii}(2\bar{p}_i + \hat{p}_i) + A_{ij}p_j + b_i)) \\
&= \pi_i^{nce} + \bar{p}_i A_{ij}\hat{p}_j + \hat{p}_i(A_{ii}(2\bar{p}_i + \hat{p}_i) + A_{ij}p_j + b_i) \\
\hat{\pi}_i &= \bar{p}_i A_{ij}\hat{p}_j + \hat{p}_i(2A_{ii}\bar{p}_i + A_{ij}\bar{p}_j + b_i + A_{ii}\hat{p}_i + A_{ij}\hat{p}_j) \\
&= A_{ii}\hat{p}_i^2 + A_{ij}p_i\hat{p}_j
\end{aligned} \tag{3.7}$$

This idea can also be extended to revenue:

$$\hat{v}_i = \hat{\pi}_i - \beta_i A_{ij} p_i \hat{p}_j + \beta_j A_{ji} p_j \hat{p}_i. \tag{3.8}$$

This small change in representation is helpful because it facilitates reasoning about whether or not a firm is better off cooperating than acting independently. If $\hat{v}_i$ is non-negative, than the firm is at least as well off, and if $\hat{v}_i$ is negative, the firm is worse off.

A necessary condition for a coalition to be natural is that the all participating firms make at least as much as they would without cooperation, which implies that the total profits in the system be at least as great as the total profits produced in the non-cooperative case. The set of price vectors $\hat{p}$ which satisfy this property can be identified geometrically. Let $\hat{w} = \sum_{i \in N} \hat{\pi}_i$ be the change in total profits in the system resultant from cooperation. The ellipse described by the expression

$$\hat{w} = \sum_{i \in N} \hat{\pi}_i = A_{ii}\hat{p}_i^2 + A_{ij}p_i\hat{p}_j + A_{jj}\hat{p}_j^2 + A_{ij}p_j\hat{p}_i = 0 \tag{3.9}$$

delineates a contour of prices for which the total profits of the system are equal to the profits of the non-cooperative system. For any point on or inside this ellipse, the total profits of the system are at least as great as the total profits of the non-cooperative system.

For the pricing policies represented by points which lie on this contour, there exists exactly one redistribution of total profits which makes both firms at least as well off as they would be without any cooperation. For all pricing policies within the ellipse, however, there exist multiple distributions which meet this requirement. We can assume that the distribution function works by giving both firms at least as much as they would have made without cooperation and then distributing the surplus $\hat{w}$ (which is always greater than zero inside the ellipse) between the two. As there are infinite ways to divide the surplus, there are infinite distribution policies which satisfy the requirements of a natural coalition.

Perhaps because of the numerous possible ways to distribute cooperative gains, describing rules for side payments via β dynamics proves less straightforward than describing price dynamics. Taking the partial derivative of revenues with respect to β_i yields

$$\dot{\beta}_i = \frac{\partial v_i}{\partial \beta_i} = -A_{ij}\hat{p}_j p_i. \tag{3.10}$$

This approach to defining dynamics for β_i, however, does not capture the feedback effects of side payments and consequently results in no cooperation. When the product $A_{ij}\hat{p}_j$ is positive, it indicates that firm j has moved its price in a manner favorable to firm i, producing a windfall of $A_{ij}p_i\hat{p}_j$ for that firm. Note that when this is the case, the time derivative of β_i is negative, indicating that firm i wants to decrease β_i, the percentage of the windfall it gives to firm j. In simulations, this resulted in all β variables decreasing until capped at 0, which results in no cooperation and no deviation from the NCE.

The failure of the gradient approach requires the consideration of other sets of dynamics. These must still be selfishly motivated, that is they must still try to maximize the individual firm's profits, but they must also take into account the feedback effects of side payments. It should be noted that a number of different rules for side payments result in natural coalitions, each with a slightly different distribution of profits. Here, we present a specific rule.

To understand the feedback effects of a choice of β_i, consider the equilibrium value $\hat{p}_j^{eq}$ when all other variables are held constant. Note that symmetry in firms allows for the interchange of subscripts i and j. From $\dot{p}_j = 0$ and (3.6),

$$\hat{p}_j^{eq} = \frac{(1-\beta_j)A_{ji}\hat{p}_i + \beta_i A_{ij} p_i}{-2A_{jj}}. \tag{3.11}$$

The partial derivative of $\hat{p}_j^{eq}$ can be used to approximate the feedback effect of a choice of side payment coefficient β_i:

$$\frac{\partial \hat{p}_j^{eq}}{\partial \beta_i} = \frac{A_{ij} p_i}{-2A_{ii}}. \tag{3.12}$$

If we then substitute $\hat{p}_j^{eq}$ into $\hat{v}_i$ for $\hat{p}_j$ and take the partial derivative of this new expression with respect to β_i results in the following dynamics:

$$\dot{\beta}_i = -A_{ij}\hat{p}_j p_i + (1-\beta_i)\frac{(A_{ij}p_i)^2}{-2A_{ii}} + \beta_j A_{ji} p_j \hat{p}_i. \tag{3.13}$$

However, dropping the last term results in negligible difference in equilibrium prices, profits and revenues. Because of this and because the simplified beta dynamics greatly reduce the complexity of analysis, beta dynamics are finally defined as follows:

$$\dot{\beta}_i = -A_{ij}\hat{p}_j p_i + (1-\beta_i)\frac{(A_{ij}p_i)^2}{-2A_{ii}}. \tag{3.14}$$

The desire to maximize a firm's own profits by reducing the amount given in a side payment is counteracted by the second term, which can be thought of as a recognition of the need to incentivize the other firm appropriately so that it shifts its price favorably. Having established system dynamics we now investigate the properties of the system at equilibrium.

3.2 Properties at Equilibrium

One of the chief motivation for designing this system of side payments was to guarantee that all firms would benefit from cooperation. Here, we show that at equilibrium, all cooperative coalitions are natural coalitions, or in other words, $\forall i \in N$, $\hat{v}_i^{ce} \geq 0$. Recall from (3.8) that $\hat{v}_i \geq 0$ implies $v_i \geq \pi_i^{nce}$. We now show this is true for the dynamics given in (3.6) and (3.14).

Lemma 3.1. *If $sign(\hat{p}_i) \neq sign(A_{ji})$, then $\beta_j = 1$.*

Proof. At equilibrium,

$$\begin{aligned} \beta_j &= \frac{(2A_{ii}\hat{p}_i + A_{ji}p_j)}{(A_{ji}p_j)} \\ &= \frac{2A_{ii}\hat{p}_i}{A_{ji}p_j} + 1. \end{aligned} \tag{3.15}$$

Since $A_{jj} < 0$ and $sign(A_{ij}) \neq sign(\hat{p}_j)$, all terms are positive, and the raw value of β_j is greater than 1. However, explicit constraints demand that $0 \leq \beta_j \leq 1$, which caps β_j at 1. □

Theorem 3.2. *Given an admissible linear demand function, system dynamics described in (3.6) and (3.14) result in natural coalitions.*

Proof. This is proved by showing that $\forall i \in N$, $\hat{v}_i^{ce} \geq 0$. The expression for $\hat{v}_i$ is quadratic in $\hat{p}_i$, and can be rewritten as $\hat{v}_i = a\hat{p}_j^2 + b\hat{p}_j + c$.

$$\hat{v}_i = A_{ii}\hat{p}_i^2 + ((1-\beta_i)\hat{p}_j A_{ij} + \beta_j p_j A_{ji})\hat{p}_i + (1-\beta_i)\hat{p}_j A_{ij}\bar{p}_i. \tag{3.16}$$

Because A_{ii} is negative, this expression represents a concave parabola (holding all variables other than $\hat{p}_i$ constant), which indicates that the vertex will be a maximum and will always be greater than or equal to the $\hat{v}_i$-axis intercept $(1-\beta_i)\hat{p}_j A_{ij}\bar{p}_i$. As the dynamics of $\hat{p}_i$ given in (3.6) maximize the above expression, if the value of the intercept is greater than or equal to 0, then the net effect of cooperation on firm i is non-negative.

Consider two cases: $sign(A_{ij}) = sign(\hat{p}_j)$ and $sign(A_{ij}) \neq sign(\hat{p}_j)$. In the first case, examining the signs of the factors shows that $(1-\beta_i)\hat{p}_j A_{ij}\bar{p}_i \geq 0$. Explicit limits on $\bar{p}_i$ and β_i, ensure that $\bar{p}_i$ and $(1-\beta_i)$ are non-negative. Furthermore, because $sign(A_{ij}) = sign(\hat{p}_j)$, the product of these two is also non-negative, and the entire term must be non-negative at equilibrium.

In the second case, $sign(A_{ij}) \neq sign(\hat{p}_j)$, which by lemma (3.1) implies β_i will be 1. As a result, $(1-\beta_i)\hat{p}_j A_{ij}\bar{p}_i = 0$. Therefore, the dynamics always result in all firms being at least as well off as they would be under non-cooperative dynamics. □

From this, it also clearly follows that the producer surplus of the system is at least as great as it was at NCE. Natural coalitions are guaranteed. Furthermore, it can be shown that for any admissible linear demand function which has non-zero cross-price elasticity, each firm is strictly better off than it was at the NCE.

Lemma 3.3. *At equilibrium, if $\hat{p}_i = 0$, then $\beta_j = 1$.*

Proof. This follows directly from substitution into (3.15). □

Lemma 3.4. *Given a linear demand function and the the dynamics in (3.6) and (3.14), at equilibrium, $(\hat{p}_i = \hat{p}_j = 0) \leftrightarrow (A_{ij} = A_{ji} = 0)$.*

Proof. First, we will prove that if at equilibrium $\hat{p}_i = \hat{p}_j = 0$, then $A_{ij} = A_{ji} = 0$. We proceed by contradiction, assuming that at equilibrium, $\hat{p}_i = \hat{p}_j = 0$ and $A_{ij}, A_{ji} \neq 0$. From lemma (3.3), $\beta_i = \beta_j = 1$. Substituting into (3.11),

$$\hat{p}_j = 0 = \frac{\bar{p}_i A_{ij}}{-2A_{ii}}, \tag{3.17}$$

which could only be true given an admissible linear demand if $A_{ij} = 0$. This is a contradiction. Therefore $(\hat{p}_i = \hat{p}_j = 0) \implies (A_{ij} = A_{ji} = 0)$.

The assertion that if $A_{ij} = A_{ji} = 0$, then $\hat{p}_i = \hat{p}_j = 0$, follows directly by substitution into (3.11). $\square$

This lemma shows that so long as cross-price elasticity of demand in the market is non-zero, cooperation will occur. We now proceed to show that this cooperation always results in both firms being strictly better off than they would be at the NCE.

Theorem 3.5. *Given an admissible linear demand function which demonstrates cross elasticity of demand, the equilibrium produced by the proposed system dynamics is such that both firms are better off than they would be at the NCE.*

Proof. The expression for $\hat{v}_i$ in (3.16) can be rewritten in vertex form

$$\hat{v}_i = A_{ii}\left(\hat{p}_i - \frac{((1-\beta_i)A_{ij}\hat{p}_j + \beta_j A_{ji}p_j)}{-2a_{ii}}\right)^2 + (1-\beta_i)\hat{p}_j A_{ij}\bar{p}_i - \frac{((1-\beta_i)\hat{p}_j A_{ij} + \beta_j p_j A_{ji})^2}{4A_{ii}}. \tag{3.18}$$

At equilibrium then, $\hat{v}_i = (1-\beta_i)\hat{p}_j A_{ij}\bar{p}_i - \frac{((1-\beta_i)\hat{p}_j A_{ij} + \beta_j p_j A_{ji})^2}{4A_{ii}}$, and it can be shown that this value is always positive. Again, consider two cases: $sign(A_{ij}) = sign(\hat{p}_j)$ and $sign(A_{ij}) \neq sign(\hat{p}_j)$. In the first case, all terms are positive, so $\hat{v}_i > 0$. In the second case, $\beta_i = 1$, so again $\hat{v}_i > 0$. $\square$

In summary, the system demonstrates the desired properties at equilibrium. When corss-price elasticity of demand in the market is non-zero, cooperation induced through side payments makes both participants strictly better off.

3.2.1 Stability

This section shows that equilibria of the system are locally stable, meaning that, given time and the correct initial conditions, firms will settle into natural coalitions. The dynamics for the system with side payments are non-linear. To facilitate analysis, consider the system dynamics obtained by linearizing the system about an equilibrium point $q = (p_i^{ce}, p_j^{ce}, \beta_i^{ce}, \beta_j^{ce})$. The superscript ce indicates the system is at the cooperative equilibrium. Perturbations around the equilibrium point are denoted with Δ, so that Δp_i represents a small perturbation in p_i. Linearized dynamics are obtained by taking the Taylor series expansion of the dynamics around the equilibrium point and keeping only the first two terms. The higher order terms are ignored.

$$\begin{aligned}
\dot{\hat{p}}_i = \dot{p}_i = f_1(\hat{p}_i, \hat{p}_j, \beta_i, \beta_j) &= 2A_{ii}p_i + A_{ij}p_j + b_i + (1-\beta_i)A_{ij}\hat{p}_j + \beta_j A_{ji} p_j \\
&= f_1(q) + \left.\frac{\partial f_1}{\partial p_i}\right|_q \Delta\hat{p}_i + \left.\frac{\partial f_1}{\partial p_j}\right|_q \Delta\hat{p}_j + \left.\frac{\partial f_1}{\partial \beta_i}\right|_q \Delta\beta_i + \left.\frac{\partial f_1}{\partial \beta_j}\right|_q \Delta\beta_j + higher\ order\ terms \\
\dot{\beta}_i = f_2(\hat{p}_i, \hat{p}_j, \beta_i, \beta_j) &= -A_{ij}\hat{p}_j p_i + (1-\beta_i)A_{ij}p_i \frac{A_{ij}p_i}{-2A_{jj}} \\
&= f_2(q) + \left.\frac{\partial f_2}{\partial p_i}\right|_q \Delta\hat{p}_i + \left.\frac{\partial f_2}{\partial p_j}\right|_q \Delta\hat{p}_j + \left.\frac{\partial f_2}{\partial \beta_i}\right|_q \Delta\beta_i + \left.\frac{\partial f_2}{\partial \beta_j}\right|_q \Delta\beta_j + higher\ order\ terms
\end{aligned} \tag{3.19}$$

The linearized system then becomes

$$\begin{bmatrix} \dot{\Delta\hat{p}} \\ \dot{\Delta\beta} \end{bmatrix} = L \begin{bmatrix} \Delta\hat{p} \\ \Delta\beta \end{bmatrix}$$

$$L = \begin{bmatrix} 2A_{ii} & (1-\beta_i^{ce})A_{ij} + \beta_j^{ce}A_{ji} & -A_{ij}\hat{p}_j^{ce} & A_{ji}p_j^{ce} \\ (1-\beta_j^{ce})A_{ji} + \beta_i^{ce}A_{ij} & 2A_{jj} & A_{ij}p_i^{ce} & -A_{ji}\hat{p}_i^{ce} \\ -\hat{p}_j^{ce}A_{ij} + \frac{2(1-\beta_i^{ce})A_{ij}^2 p_i^{ce}}{-2A_{jj}} & -A_{ij}p_i^{ce} & \frac{(A_{ij}p_i^{ce})^2}{2A_{jj}} & 0 \\ -A_{ji}p_j^{ce} & -\hat{p}_i^{ce}A_{ji} + \frac{2(1-\beta_j^{ce})A_{ji}^2 p_j^{ce}}{-2A_{ii}} & 0 & \frac{(A_{ji}p_j^{ce})^2}{2A_{ii}} \end{bmatrix} \tag{3.20}$$

Note that L_{31} and L_{42} can be rewritten by substituting in $\beta_i^{ce} = \frac{2A_{jj}\hat{p}_j^{ce}}{A_{ij}p_i^{ce}} + 1$ and $\beta_j^{ce} = \frac{2A_{ii}\hat{p}_i^{ce}}{A_{ji}p_j^{ce}} + 1$, and algebraically simplifying. This results in the a simplified A

matrix.

$$L = \begin{bmatrix} 2A_{ii} & (1-\beta_i^{ce})A_{ij} + \beta_j^{ce}A_{ji} & -A_{ij}\hat{p}_j^{ce} & A_{ji}p_j^{ce} \\ (1-\beta_j^{ce})A_{ji} + \beta_i^{ce}A_{ij} & 2A_{jj} & A_{ij}p_i^{ce} & -A_{ji}\hat{p}_i^{ce} \\ A_{ij}\hat{p}_j^{ce} & -A_{ij}p_i^{ce} & \frac{(A_{ij}p_i^{ce})^2}{2A_{jj}} & 0 \\ -A_{ji}p_j^{ce} & A_{ji}\hat{p}_i^{ce} & 0 & \frac{(A_{ji}p_j^{ce})^2}{2A_{ii}} \end{bmatrix}. \quad (3.21)$$

Lemmas concerning numerical range can be leveraged to show that the linearized system is stable. For $A \in \mathbb{C}^{n\times n}$, numerical range is denoted $W(A) = \{x^*Ax \mid ||x||_2 = 1\}$, and the spectrum of A is denoted $\sigma(A)$. From [7],

Lemma 3.6. *For complex valued square matrices A and B:*

1. $W(A)$ is compact and convex.
2. $\sigma(A) \subset W(A)$.
3. $W(A+B) \subset W(A) + W(B)$.

Lemma 3.7. *Given L in (3.20), the matrix decomposes to $L = B + C$ where B is*

$$B = \begin{bmatrix} 2A_{ii} & (1-\beta_i^{ce})A_{ij} + \beta_j^{ce}A_{ji} & 0 & 0 \\ (1-\beta_j^{ce})A_{ji} + \beta_i^{ce}A_{ij} & 2A_{jj} & 0 & 0 \\ 0 & 0 & \frac{(A_{ij}p_i^{ce})^2}{2A_{jj}} & 0 \\ 0 & 0 & 0 & \frac{(A_{ji}p_j^{ce})^2}{2A_{ii}} \end{bmatrix} \quad (3.22)$$

and C is

$$C = \begin{bmatrix} 0 & 0 & -A_{ij}\hat{p}_j^{ce} & A_{ji}p_j^{ce} \\ 0 & 0 & A_{ij}p_i^{ce} & -A_{ji}\hat{p}_i^{ce} \\ A_{ij}\hat{p}_j^{ce} & -A_{ij}p_i^{ce} & 0 & 0 \\ -A_{ji}p_j^{ce} & A_{ji}\hat{p}_i^{ce} & 0 & 0 \end{bmatrix} \quad (3.23)$$

Lemma 3.8. *Given B in (3.22) and an admissible linear demand function $q(p) = Ax + b$, $max\,\mathrm{Re}(W(B)) < 0$.*

Proof. The proof proceeds by showing that B is negative definite. Note that due to the restrictions on A imposed by the assertion that $q(p)$ is admissible and the fact that any β value must be between 0 and 1, B is diagonally dominant. Furthermore, all diagonal entries are negative, which indicates that B is a negative definite matrix.

□

Lemma 3.9. *Given C in (3.23), $\mathrm{Re}\,(W(C)) = 0$.*

Proof. Algebraically, the real part of numerical range x^*Cx for the matrix is

$$\begin{aligned}\mathrm{Re}\,(x^*Cx) &= \mathrm{Re}\,(-A_{ij}\hat{p}_j^{ce}x_3x_1^* + A_{ji}p_j^{ce}x_4x_1^* + A_{ij}p_i^{ce}x_3x_2^* - A_{ji}\hat{p}_i^{ce}x_4x_2^* \\ &\quad + A_{ij}\hat{p}_j^{ce}x_1x_3^* - A_{ji}p_j^{ce}x_1x_4^* - A_{ij}p_i^{ce}x_2x_3^* + A_{ji}\hat{p}_i^{ce}x_4x_2^*) \\ &= \mathrm{Re}\,(x_3x_1^*)(-A_{ij}\hat{p}_j^{ce} + A_{ij}\hat{p}_j^{ce}) + \mathrm{Re}\,(x_4x_1^*)(A_{ji}p_j^{ce} - A_{ji}p_j^{ce}) \\ &\quad + \mathrm{Re}\,(x_3x_2^*)(A_{ij}p_i^{ce} - A_{ij}p_i^{ce}) + \mathrm{Re}\,(x_4x_2^*)(-A_{ji}\hat{p}_i^{ce} + A_{ji}\hat{p}_i^{ce}) \\ &= 0\end{aligned} \tag{3.24}$$

□

Theorem 3.10. *Given an admissible linear demand function, any equilibrium of the system with side payments given by (3.19) is locally stable.*

Proof. The proof proceeds using a numerical range argument similar to that used in [4] and [3]. From lemma (3.7), the dynamics matrix for the linearized system can be rewritten as $L = B + C$. From lemma (3.6), $W(L) = W(B + C) \subset (W(B) \cup W(C))$. Therefore,

$$max\,\mathrm{Re}(W(L)) \leq max\,\mathrm{Re}(W(B)) + max\,\mathrm{Re}(W(C))$$

From lemmas (3.8) (3.9), $\mathrm{Re}(W(B)) < 0$ and $\mathrm{Re}(W(C)) = 0$. Therefore,

$$max\,\mathrm{Re}(W(L)) < 0,$$

and applying lemma (3.6),

$$max\,\mathrm{Re}(\sigma(L)) \leq max\,\mathrm{Re}(W(L)) < 0$$

. □

Thus any equilibrium of the system is locally stable. This, however, presents two pressing questions. Firstly, where are those equilibria, and secondly, what is basin of attraction for each? These are important questions because although we have shown properties of equilibria generically, we have yet to prove that any equilibrium exists. Furthermore, this analysis does not preclude the existence of an equilibrium in which the value of either β is not in the required range.

3.3 Numerical Examples

This issue has proven to be a particularly difficult nut to crack. However, multiple numerical examples all show the existence of a reasonable equilibrium with a basin of attraction that appears to cover all sensible initial conditions. To prove this point, several numerical examples are provided. For each example, the system was simulated from six-hundred twenty-five distinct initial conditions distributed evenly throughout the space of reasonable initial conditions.

3.3.1 Numerical Example 1 (Complements)

Consider a system defined by the following admissible linear demand function:

$$q(p) = \begin{bmatrix} -3 & -2 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \begin{bmatrix} 6 \\ 6 \end{bmatrix}. \quad (3.25)$$

The results of simulating this system with non-cooperative, perfectly cooperative, and dynamic coalition formation dynamics are shown in the table below.

Dynamics	$\bar{p}_1$(\$)	$\hat{p}_1$(\$)	p_1(\$)	$\bar{p}_2$(\$)	$\hat{p}_2$(\$)	p_2(\$)	β_1	β_2	v_1(\$)	v_2(\$)	w(\$)
Non-Cooperative	0.71	0	0.71	0.88	0	0.88	0	0	1.49	2.34	3.83
Perfectly Cooperative	0.71	-0.04	0.67	0.88	-0.21	0.67	0	0	1.78	2.22	4.00
Dynamic Coalition	0.71	-0.05	0.66	0.88	-0.11	0.78	0.50	0.50	1.58	2.39	3.97

TABLE 3.1: System variables at equilibrium for a two firm market selling complementary goods

As expected, the total producer surplus under dynamic coalition dynamics is much closer than producer surplus under non-cooperative dynamics to the maximum producer surplus. Although perfect cooperation generates higher producer surplus, it is distributed in such a way that the second firm is better off not cooperating. Instead, the second firm would likely defect. Under dynamic coalition dynamics however, both firms are better off. Also, the equilibrium point found by simulating the system is reasonable, as both β's are between 0 and 1 and prices are positive.

3.3.2 Numerical Example 2 (Substitutes)

Consider the same function given in Example 1, but with substitute rather than complementary goods:

$$q(p) = \begin{bmatrix} -3 & 2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \begin{bmatrix} 6 \\ 6 \end{bmatrix}. \tag{3.26}$$

The results of simulating this system with non-cooperative, perfectly cooperative, and dynamic coalition formation dynamics are shown in the table below.

Dynamics	$\bar{p}_1$(\$)	$\hat{p}_1$(\$)	p_1(\$)	$\bar{p}_2$(\$)	$\hat{p}_2$(\$)	p_2(\$)	β_1	β_2	v_1(\$)	v_2(\$)	w(\$)
Non-Cooperative	1.41	0	1.41	1.24	0	1.24	0	0	5.98	4.58	10.56
Perfectly Cooperative	1.41	0.59	2.00	1.24	0.76	2.00	0	0	8.00	4.00	12.00
Dynamic Coalition	1.41	0.15	1.56	1.24	0.27	1.50	0.48	0.41	6.43	4.90	11.33

Table 3.2: System variables at equilibrium for a two firm market selling substitute goods

The table shows similar trends with regards to individual profits and social welfare as described in Example 1. This example serves to show the difference between cooperation between firms selling complementary and substitute goods. Note that total profits are much higher, and consequently, price shifts more dramatic. Also, as in the previous example, price shifts are of the same sign as the corresponding value in the A matrix, that is $sign(A_{ij}) = sign(\hat{p}_j)$. This indicates that at equilibrium, both firms have deviated from their non-cooperative prices in a way which benefits the other firm. This proved to be the case in all simulations.

3.4 Conclusion

This chapter has presented an altered representation of the market used in the gradient play differential game representing cooperation in a competitive environment. This change facilitates reasoning about revenues with respect to the revenues at the NCE and explicitly represents side payments. The specific set of dynamics for the evolution of price and side payments presented was shown to result in natural coalition at equilibrium, and the local stability of any equilibrium was proven. Although proofs of existence of a reasonable equilibrium, many simulations show that it does in fact exist and that its basin of attraction covers all reasonable initial conditions.

Chapter 4

Conclusion and Future Work

This work introduced a gradient play differential game played in a producer market as a model for behavior in a competitive environment. The idea of cooperation in this framework was introduced, aided by definitions of coalitions and natural coalitions. Chapter 2 showed that existing representation of cooperation within this framework produce coalitions but do not guarantee that these coalitions are natural. Chapter 3 presented system dynamics which induce natural coalitions in a two firm market. Mathematical proofs show that, given an admissible linear demand function, any equilibrium of the system is locally stable and that any equilibrium results in both firms being at least as well of as they would be acting independently. Furthermore, if the demand function demonstrates cross elasticity of demand, then cooperation in the form of side payments always occurs and both firms are guaranteed to be strictly better off. Although empirical evidence suggests it is true, proving mathematically the existence of an admissible equilibrium with a region of attraction that includes all admissible initial conditions remains a pressing problem. Also, these dynamics should be generalized to cover not just two-firm markets but markets with any number of firms.

The dynamics presented here are not the only possible dynamics which would cause the system to behave as desired, and throughout the research process, slight variations were shown to behave in similar ways. Importantly, different dynamics appear to lead to different distribution at equilibrium. Understanding the distribution bias a set of dynamics implies in a structured way is an important future step towards building appropriate models of cooperation as well as inducing cooperation in otherwise non-cooperative environments.

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