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Cooperation Induction in Two Player Bertand Markets with Linear Demand

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***Abstract*—We present a two player gradient play differential game in a producer market with quadratic payoff as a model of behavior in a competitive environment and show that the Nash equilibrium is not efficient. An algorithm is then presented which uses side payments to induce cooperation between firms, and rules for side payment strategies are shown. The stability of the new system at a reasonable equilibrium is proved, and it is shown that all participants are at least as well off as they would be at the non-cooperative equilibrium. Numerical examples show the existence of a reasonable equilibrium and that the basin of attraction of that equilibrium appears to cover all reasonable initial conditions.**

Keywords*—*cooperation, game theory.

I. Introduction

Crossover between game theory and control theory is common, and the theoretical framework game theory provides of utility maximizing agents has been applied to reason about and create decentralized control in a number of contexts. Specific applications include distributed control in power grids [1], [2] and urban drainage systems [3]. In [4], the authors discuss the challenges facing coalition control, which leverages myopic utility maximization by individual components to achieve a large scale objective.

In a related vein, we consider the problem of inducing cooperative behavior in a competitive environment to achieve a system-wide objective. Questions relating to the cooperation of rational agents in market-like situations traditionally belong to the purview of cooperative game theory. In this context, this system would be modeled as a coalitional game with transferable payoffs. However, this representation fails to capture the effects of one coalition's choices on the payoffs of other coalitions. The characteristic function of cooperative games maps any subset of the agents to some payoff for that coalition, frequently ignoring the coalitional structure and strategy of the agents not participating in the firm in question [5]. Though variations which attempt to take this into account exist [6], a systems and control perspective provides another useful angle on the problem.

Certain representations of firms' behaviors in markets, especially gradient play differential games, lend themselves to control and systems analysis. In [7] and [8], the behavior of firms which completely participate in exactly one coalition is examined and conditions for the existence of a single, globally stable equilibrium are provided. Similar results are found in [9], which extends analysis to permit firms to participate in multiple, overlapping coalitions. In these works, coalition structure is given *a priori*, which can lead to individual firms being worse off because of cooperative behavior.

Although these cases may imply a redistribution mechanism, it is never provided explicitly. We are interested in formally describing the dynamics of these redistributive side payments. What side payment dynamics will drive a system to maximize welfare? How do these dynamics affect the distribution of cooperative gains amongst collaborators? Better understanding answers to these questions will allow us to both design and reason about systems in which agents cooperate and dynamically allocate their cooperative gains.

This paper takes first steps towards better understanding the dynamics of side payments and contributes to existing literature by providing an explicit mechanism for transferring utility (money) which successfully induces cooperation in a two player system without the arbitrary imposition of *a priori* cooperative structure. Although all players act for strictly selfish reasons, the explicit side payment mechanism leads selfish players to cooperate.

The following section presents the non-cooperative producer market model and mathematical definitions. Section III introduces a mechanism designed to induce cooperation among selfish agents through side payments. Subsequently, Section IV proves local stability and other properties of reasonable equilibria, and Section V provides a numerical example which empirically demonstrates the existence of a reasonable equilibrium and a basin of attraction including all reasonable initial conditions. Finally, Section VI concludes the paper with a brief summary and direction of future work.

II. Two-Firm Bertrand Producer Market

A. Preliminaries

We consider a two-firm Bertrand producer market. Let $N = \{1, 2\}$ be the set of firms in the market. The market includes a price vector $p \in \mathbb{R}^2$, a demand function $q(p) : \mathbb{R}^2 \mapsto \mathbb{R}^2$ and profit function $\pi(p) : \mathbb{R}^2 \mapsto \mathbb{R}^2$. The price, demand and profits associated with a specific firm $i \in N$ are denoted with a subscript: $p_i \in \mathbb{R}$, $q_i(p) : \mathbb{R}^2 \mapsto \mathbb{R}$ and $\pi_i(p) : \mathbb{R}^2 \mapsto \mathbb{R}$. Firms are symmetrical, so that subscripts can be interchanged. Each firm sells an ordinary, differentiable good at a price p_i which the firm determines.

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Cross-price demand may exist in the market, meaning that the demand for each good is a function of all prices. All firms are assumed to produce at zero cost, so profits are given by the product of price and demand:

$$\pi_i = p_i q_i(p). \quad (1)$$

While individual profits tell us how each firm fares, the sum of all profits measures producer performance as a whole.

Definition 1. *The sum over all profits in the market is called Producer Surplus, w:*

$$w(p) = \sum_{i \in N} \pi_i(p). \quad (2)$$

Definition 2. *The Maximum Producer Surplus or Cartel Outcome w^* is the maximum producer surplus the market can achieve over all possible price combinations when such a maximum exists, otherwise the maximum producer surplus is understood to be infinity:*

$$w^* = \max_p w(p). \quad (3)$$

B. Linear Demand and Optimal Pricing Policy

Definition 3. *An Optimal Pricing Policy p^* is a set of prices which result in the market producing maximum producer surplus:*

$$p^* = \underset{p}{\text{argmax}}\ w(p). \quad (4)$$

Note that depending on the demand function, a market may have multiple optimal pricing policies, all of which produce the same producer surplus. This paper considers linear demand functions, which have the form

$$q(p) = Ap + b, \quad A \in \mathbb{R}^{2\times 2}, b \in \mathbb{R}^2. \quad (5)$$

Entry a_{ij} represents the sensitivity of demand for good i with respect to changes in price of good j. This sensitivity of demand will be positive for substitute goods and negative for complementary goods, and the relationship between goods as complements or substitutes is symmetric. Because the goods sold are assumed to be normal, an increase in a good's price always decreases the demand for that good, so diagonal entries must be negative. Additionally, a firm's own price affects its demand more than the combined influence of all other goods, and affects its own demand more than it affects all other goods' demands combined. These restrictions give rise to the following definition:

Definition 4. *A linear demand function $q(p) = Ap + b$ is said to be admissible if the following conditions are met:*

1) *A and A^T are diagonally dominant.*
2) *A is sign symmetric.*
3) *$\forall i \in N$, $a_{ii} < 0$.*
4) *$\forall i \in N$, $b_i > 0$.*

The following definition relating to the properties of demand functions will also be useful:

Definition 5. *A linear demand function $q(p) = Ap + b$ is said to have non-zero cross-price elasticity of demand if A is not diagonal.*

The requirement of diagonal dominance on both A and A^T is necessary to ensure that the market converges to a stable equilibrium under gradient play and sufficient to ensure that A and $(A + A^T)$ are invertible. This in turn ensures the existence of exactly one optimal pricing policy, which can be found analytically. Flor $p \in \mathbb{R}^n$, let $diag(p) = D \in R^{n\times n}$ denote a diagonal matrix where $d_{ii} = p_i$. Let $\mathbf{1}$ indicate an appropriately sized vector of 1's. Producer surplus may be written as

$$w = \mathbf{1}^T\ \Pi(p) = \mathbf{1}^T diag(p) q(p) = p^T(Ap + b). \quad (6)$$

The maximum is found by taking the derivative with respect to p and setting that expression equal to 0:

$$\frac{\partial w}{\partial p} = (A + A^T)p + b = 0. \quad (7)$$

Because $(A+A^T)$ is invertible, there is exactly one optimal pricing policy given by $p^* = -(A + A^T)^{-1} b$, which is

$$p^* = \frac{-1}{4a_{11}a_{22} - (a_{12} + a_{21})^2} \begin{bmatrix} 2a_{22}b_1 - (a_{12} + a_{21})b_2 \\ 2a_{11}b_2 - (a_{12} + a_{21})b_1 \end{bmatrix}. \quad (8)$$

Therefore, any market characterized by an admissible linear demand function has exactly one optimal pricing policy, and any other pricing policy leaves unrealized profits in the system from the point of view of the producers. Analysis of consumer surplus is ignored here but will be considered in future work.

C. Non-cooperative Nash Equilibrium

A gradient play differential game can represent the behavior of rational agents in this environment. In gradient play differential games, each firm's strategy is defined by the gradient of its payoff function [10]. In the current context, a firm's strategy is the price set for the corresponding good, and its strategy set is restricted to positive real numbers. With no cooperation, the payoff of each firm equals the firm's profit. This gives rise to the following differential equation which represents a firm's pricing decisions:

$$\dot{p}_i = \frac{\partial \pi_i}{\partial p_i} = q_i(p) + p_i \frac{\partial q_i}{\partial p_i}. \quad (9)$$

Here and throughout the rest of this work, ' $\cdot$ ' indicates a time derivative of a variable. Dynamics for the entire system are then:

$$\dot{p} = (A + \hat{A})p + b. \quad (10)$$

where $\hat{a}_{ii} = a_{ii}$ and $\hat{a}_{ij} = 0$. The stability of the system described by (10) is deduced by noting that since A and $\hat{A}$ are diagonally dominant, they are also Hurwitz, and their sum, $(A + \hat{A})$, is Hurwitz as well. Furthermore, this guarantees a single, unique equilibrium. Hence the restrictions put on A are necessary to ensure the system's stability and sufficient to ensure the uniqueness of the equilibrium. At this non-cooperative equilibrium (NCE), each firm's price is

$$p^{nce} = \frac{-1}{4a_{11}a_{22} - (a_{12} + a_{21})^2} \begin{bmatrix} 2a_{22}b_1 - a_{12}b_2 \\ 2a_{11}b_2 - a_{21}b_1 \end{bmatrix}. \quad (11)$$

The NCE is a Nash equilibrium, as any unilateral price change results in a loss of profits for the firm that deviates.

We now compare the optimal pricing strategy with the prices in market at the NCE. For an admissible linear demand function with non-zero cross-price elasticity of demand, $p^{nce} \neq p^*$, which implies that $w^{nce} < w^*$. However, if the cross-price elasticity is zero, then the $p^{nce} = p^*$ and $w^{nce} = w^*$. This indicates that there is no benefit to cooperation if there is zero cross-price elasticity of demand, and in subsequent analysis we consider markets where cross-price elasticity is non-zero.

In general, side payments can be represented as a portion of players' profits which will transfer to another player. This results in a single Pareto pricing policy at p^* because if $p \neq p^*$, then there exists a distribution of producer welfare at p^* such that both players are at least as well off and at least one player is better off. Therefore, only the optimal pricing policy is Pareto optimal.

III. Market With Side Payments

For dynamic coalition formation, we decompose the price variables to facilitate reasoning about effects of co-operation. Let price now be given by $p = \bar{p} + \hat{p}$. The first set of new price variables, $\bar{p}$, represent the price that the firm would set if it acted completely independently while the second, $\hat{p}$ represent a deviation from the non-cooperative price due to cooperation. The expression for an individual's profits under the new basis becomes

$$\begin{aligned}\pi_i = p_i q_i(p) &= (\bar{p}_i + \hat{p}_i)(A_{ii}(\bar{p}_i + \hat{p}_i) + A_{ij}(\bar{p}_j + \hat{p}_j) + b_i) \\ &= (\bar{p}_i + \hat{p}_i)(A_{ii}(\bar{p}_i + \hat{p}_i) + A_{ij}\bar{p}_j + b_i) + (\bar{p}_i + \hat{p}_i)A_{ij}\hat{p}_j.\end{aligned} \tag{12}$$

The above decomposition shows that the deviation of firm j from its non-cooperative strategy $\hat{p}_j$, results in a change in demand for the good produced by firm i equal to $A_{ij}\hat{p}_j$. This in turn changes firm i's profits by an amount equal to $p_i A_{ij}\hat{p}_j$. A side payment from a firm i to a firm j is represented as a percentage β of firm i's profits which occur because of the cooperative shift $\hat{p}_j$ of firm j:

$$\beta_i p_i A_{ij}\hat{p}_j, \quad 0 \leq \beta \leq 1. \tag{13}$$

Under this formulation, a firm's total revenue becomes the sum of profits, side payments given, and side payments received:

$$v_i = \pi_i - \beta_i A_{ij} p_i \hat{p}_j + \beta_j A_{ji} p_j \hat{p}_i. \tag{14}$$

A. Price Dynamics

We now describe a formulation for price dynamics which allows us to reason about the system with respect to NCE. Note that, holding $\bar{p}_i$ constant, $\frac{\partial \hat{p}_i}{\partial p_i} = 1$. As before, we take the partial derivative of revenue with respect to price:

$$\begin{aligned}\dot{p}_i =& \frac{\partial v_i}{\partial p_i} = 2A_{ii}p_i + A_{ij}p_j + b_i - \beta_i A_{ij}\hat{p}_j + \beta_j A_{ji} p_j \\ =& 2A_{ii}\bar{p}_i + A_{ij}\bar{p}_j + b_i \\ &+ 2A_{ii}\hat{p}_i + A_{ij}\hat{p}_j - \beta_i A_{ij}\hat{p}_j + \beta_j A_{ji} p_j.\end{aligned} \tag{15}$$

Let

$$\dot{\bar{p}}_i = 2A_{ii}\bar{p}_i + A_{ij}\bar{p}_j + b_i. \tag{16}$$

Since $p_i = \bar{p}_i + \hat{p}_i$, $\dot{p}_i = \dot{\bar{p}}_i + \dot{\hat{p}}_i$, and

$$\begin{aligned}\dot{\hat{p}}_i &= 2A_{ii}\hat{p}_i + A_{ij}\hat{p}_j - \beta_i A_{ij}\hat{p}_j + \beta_j A_{ji} p_j \\ &= 2A_{ii}\hat{p}_i + (1 - \beta_i)A_{ij}\hat{p}_j + \beta_j A_{ji} p_j.\end{aligned} \tag{17}$$

The definition of $\dot{\bar{p}}_i$ in (16) parallels the definition of non-cooperative price dynamics given in (10). Because $\bar{p}$ variables is completely decoupled from the other variables in the system, $\bar{p}$ in this cooperative system behaves exactly as p does in the non-cooperative system. Earlier discussion showed that, given an admissible linear demand function, there is a single, stable set of prices to which the system converges, the NCE. Since in the cooperative system, $\bar{p}_i$ evolves in the same way, it too will always converge to the NCE given an admissible linear demand function, and at the cooperative equilibrium (CE), $\bar{p}^{ce} = p^{nce}$.

Now we can reason about a firm's total payoff at the CE relative to its profits at the NCE. As $\hat{p}$ is conceptually a deviation from the NCE, let $\hat{\pi}$ represent gains or losses in profits relative to $\pi^i = \bar{p}_i(A_{ii}\bar{p}_i + A_{ij}\bar{p}_j + b_i)$. Recall that at the CE, $2A_{ii}\bar{p}_i + A_{ij}\bar{p}_j + b_i = 0$, so that at the CE

$$\begin{aligned}\pi_i &= \pi_i^{nce} + \bar{p}_i A_{ij}\hat{p}_j + \hat{p}_i(A_{ii}(2\bar{p}_i + \hat{p}_i) + A_{ij}p_j + b_i) \\ \hat{\pi}_i &= \bar{p}_i A_{ij}\hat{p}_j + \hat{p}_i(2A_{ii}\bar{p}_i + A_{ij}\bar{p}_j + b_i + A_{ii}\hat{p}_i + A_{ij}\hat{p}_j) \\ &= A_{ii}\hat{p}_i^2 + A_{ij}p_i\hat{p}_j\end{aligned} \tag{18}$$

This idea can also be extended to revenue:

$$\hat{v}_i = \hat{\pi}_i - \beta_i A_{ij} p_i \hat{p}_j + \beta_j A_{ji} p_j \hat{p}_i. \tag{19}$$

This small change in representation facilitates reasoning about whether or not a firm is better off cooperating than acting independently. If $\hat{v}_i$ is non-negative, than the firm is at least as well off, and if $\hat{v}_i$ is negative, the firm is worse off.

B. Beta Dynamics

A number of potential rules for side payments could result in all players being better off than they were at the NCE, each with a slightly different distribution of profits. Here, we present a specific rule.

Proceeding with the standard gradient play approach yields the following dynamics:

$$\dot{\beta}_i = \frac{\partial v_i}{\partial \beta_i} = -A_{ij}\hat{p}_j p_i. \tag{20}$$

This, however, does not capture the feedback effects of side payments and consequently results in no cooperation. When the product $A_{ij}\hat{p}_j$ is positive, it indicates that firm j has moved its price in a manner favorable to firm i, producing a windfall of $A_{ij}p_i\hat{p}_j$ for that firm. Note that when this is the case, the time derivative of β_i is negative, indicating that firm i will decrease β_i until capped at 0. These dynamics result in no cooperation and no deviation from the NCE.

To understand the feedback effects of a choice of β_i, we need to understand how a player's choice of β_i affects the other player's price. Although we have the relationship between β_i and $\dot{p}_j$, finding the relationship between β_i and $\dot{p}_j$ proves intractable. We can, however, approximate that relationship near an equilibrium. Consider the equilibrium value $\hat{p}_j^{eq}$ when all other variables are held constant. Note that symmetry in firms allows for the interchange of subscripts i and j. From $\dot{p}_j$ = 0 and (17),

$$\hat{p}_j^{eq} = \frac{(1 - \beta_j)A_{ji}\hat{p}_i + \beta_i A_{ij} p_i}{-2A_{jj}}. \tag{21}$$

The partial derivative of $\hat{p}_j^{eq}$ can be used to approximate the feedback effect of a choice of side payment coefficient β_i:

$$\frac{\partial \hat{p}_j^{eq}}{\partial \beta_i} = \frac{A_{ij}p_i}{-2A_{ii}}. \tag{22}$$

If we then substitute $\hat{p}_j^{eq}$ into $\hat{v}_i$ for $\hat{p}_j$ and take the partial derivative of this new expression with respect to β_i we arrive the following dynamics:

$$\dot{\beta}_i = -A_{ij}\hat{p}_j p_i + (1-\beta_i)\frac{(A_{ij}p_i)^2}{-2A_{ii}} + \beta_j A_{ji} p_j \hat{p}_i. \quad (23)$$

However, dropping the last term results in negligible difference in equilibrium prices, profits and revenues. Because of this and because the simplified beta dynamics greatly reduce the complexity of analysis, beta dynamics are finally defined as follows:

$$\dot{\beta}_i = -A_{ij}\hat{p}_j p_i + (1-\beta_i)\frac{(A_{ij}p_i)^2}{-2A_{ii}}. \quad (24)$$

The desire to maximize a firm's own profits by reducing the amount given in a side payment is counteracted by the second term, which can be thought of as a recognition of the need to incentivize the other firm appropriately so that it shifts its price favorably. Having established a mechanism to induce cooperation among selfish agents in (16) and (24), we now investigate the properties of the system at equilibrium.

IV. Properties at Equilibrium

We now consider properties of the system at a reasonable equilibrium.

Definition 6. *A reasonable equilibrium* $\epsilon = (p_i^{ce}, p_j^{ce}, \beta_i^{ce}, \beta_j^{ce})$ *is an equilibrium such that* $\forall i \in N$:

1) $p_i \geq 0$
2) $0 \leq \beta_i \leq 1$

Given a reasonable equilibrium, we can prove useful properties of the system.

A. Stability Analysis

The dynamics for the system with side payments given by (16) and (24) are non-linear. To facilitate analysis, consider the system dynamics obtained by linearizing the system about a reasonable equilibrium point $\epsilon = (p_i^{ce}, p_j^{ce}, \beta_i^{ce}, \beta_j^{ce})$. Perturbations around the equilibrium point are denoted with Δ, so that Δp_i represents a small perturbation in p_i. Linearized dynamics are obtained by taking the Taylor series expansion of the dynamics around the equilibrium point and ignoring all but the first two terms. The linearized system then becomes

$$\begin{bmatrix}\dot{\Delta\hat{p}}\\ \dot{\Delta\beta}\end{bmatrix} = L\begin{bmatrix}\Delta\hat{p}\\ \Delta\beta\end{bmatrix}$$

$$L = \begin{bmatrix}L_{11} & L_{12}\\ L_{21} & L_{22}\end{bmatrix}$$

$$L_{11} = \begin{bmatrix}2A_{ii} & (1-\beta_i^{ce})A_{ij} + \beta_j^{ce}A_{ji}\\ (1-\beta_j^{ce})A_{ji} + \beta_i^{ce}A_{ij} & 2A_{jj}\end{bmatrix}$$

$$L_{12} = \begin{bmatrix}-A_{ij}\hat{p}_j^{ce} & A_{ji}p_j^{ce}\\ A_{ij}p_i^{ce} & -A_{ji}\hat{p}_i^{ce}\end{bmatrix}$$

$$L_{21} = \begin{bmatrix}-\hat{p}_j^{ce}A_{ij} + \frac{2(1-\beta_i^{ce})A_{ij}^2 p_i^{ce}}{-2A_{jj}} & -A_{ij}p_i^{ce}\\ -A_{ji}p_j^{ce} & -\hat{p}_i^{ce}A_{ji} + \frac{2(1-\beta_j^{ce})A_{ji}^2 p_j^{ce}}{-2A_{ii}}\end{bmatrix}$$

$$L_{22} = \begin{bmatrix}\frac{(A_{ij}p_i^{ce})^2}{2A_{jj}} & 0\\ 0 & \frac{(A_{ji}p_j^{ce})^2}{2A_{ii}}\end{bmatrix} \quad (25)$$

Note that L_{21} can be simplified by substituting in $\beta_i^{ce} = \frac{2A_{jj}\hat{p}_j^{ce}}{A_{ij}p_i^{ce}} + 1$ and $\beta_j^{ce} = \frac{2A_{ii}\hat{p}_i^{ce}}{A_{ji}p_j^{ce}} + 1$, and algebraically simplifying. This results in

$$L_{21} = \begin{bmatrix}A_{ij}\hat{p}_j^{ce} & -A_{ij}p_i^{ce}\\ -A_{ji}p_j^{ce} & A_{ji}\hat{p}_i^{ce}\end{bmatrix}. \quad (26)$$

Lemmas concerning numerical range can be leveraged to show that the linearized system is stable. For $A \in \mathbb{C}^{n\times n}$, numerical range is denoted $W(A) = \{x^*Ax \mid ||x||_2 = 1\}$, and the spectrum of A is denoted $\sigma(A)$. From [11],

Lemma 1. *For complex valued square matrices A and B:*

1) $W(A)$ is compact and convex.
2) $\sigma(A) \subset W(A)$.
3) $W(A+B) \subset W(A) + W(B)$.

Lemma 2. *Given the L in (25), the matrix decomposes to $L = B + C$ where*

$$B = \begin{bmatrix}L_{11} & 0\\ 0 & L_{22}\end{bmatrix}$$

$$C = \begin{bmatrix}0 & L_{12}\\ L_{21} & 0\end{bmatrix} \quad (27)$$

Lemma 3. *Given B in (27) and an admissible linear demand function $q(p) = Ax + b$, $max\,\mathrm{Re}(W(B)) < 0$.*

Proof: The proof proceeds by showing that B is negative definite. Note that due to the restrictions on A imposed by the assertion that $q(p)$ is admissible and the fact that any β value must be between 0 and 1, B is diagonally dominant. Furthermore, all diagonal entries are negative, which indicates that B is a negative definite matrix. ∎

Lemma 4. *Given C in (27),* $\mathrm{Re}\,(W(C)) = 0$.

Proof: Algebraically, the real part of numerical range x^*Cx for the matrix is

$$\begin{aligned}&\mathrm{Re}\,(x_3x_1^*)(-A_{ij}\hat{p}_j^{ce} + A_{ij}\hat{p}_j^{ce}) + \mathrm{Re}\,(x_4x_1^*)(A_{ji}p_j^{ce} - A_{ji}p_j^{ce})\\ &+ \mathrm{Re}\,(x_3x_2^*)(A_{ij}p_i^{ce} - A_{ij}p_i^{ce}) + \mathrm{Re}\,(x_4x_2^*)(-A_{ji}\hat{p}_i^{ce} + A_{ji}\hat{p}_i^{ce})\\ &= 0\end{aligned} \quad (28)$$

∎

Theorem 1. *Given an admissible linear demand function, any reasonable equilibrium of the system with side payments is locally stable.*

Proof: The proof proceeds using a numerical range argument. From Lemma 2, the dynamics matrix for the linearized system can be rewritten as $L = B + C$. From Lemma 1, $W(L) = W(B + C) \subset (W(B) \cup W(C))$. Therefore,

$$max\,\mathrm{Re}(W(L)) \leq max\,\mathrm{Re}(W(B)) + max\,\mathrm{Re}(W(C))$$

From lemmas (3) (4), $\mathrm{Re}(W(B)) < 0$ and $\mathrm{Re}(W(C)) = 0$. Therefore,

$$max\,\mathrm{Re}(W(L)) < 0,$$

and applying lemma (1),

$$max\,\mathrm{Re}(\sigma(L)) \leq max\,\mathrm{Re}(W(L)) < 0.$$

Thus any equilibrium of the system is locally stable. ■

B. Beneficial Distribution

One of the chief motivations for designing this system of side payments was to guarantee that all firms would benefit from cooperation. Here, we show that at equilibrium, $\forall i \in N$, $\hat{v}_i^{ce} \geq 0$. Recall from (19) that $\hat{v}_i \geq 0$ implies $v_i \geq \pi_i^{nce}$. We now show this is true for the dynamics given in (17) and (24).

Lemma 5. *If $sign(\hat{p}_i) \neq sign(A_{ji})$, then $\beta_j^{ce} = 1$.*

Proof: At equilibrium,

$$\begin{aligned} \beta_j &= \frac{(2A_{ii}\hat{p}_i + A_{ji}p_j)}{(A_{ji}p_j)} \\ &= \frac{2A_{ii}\hat{p}_i}{A_{ji}p_j} + 1. \end{aligned} \tag{29}$$

Since $A_{jj} < 0$ and $sign(A_{ij}) \neq sign(\hat{p}_j)$, all terms are positive, and the raw value of β_j is greater than 1. However, explicit constraints demand that $0 \leq \beta_j \leq 1$, which caps β_j at 1. ■

Theorem 2. *Given an admissible linear demand function, system dynamics described in (17) and (24) result in all players being at least as well off at any reasonable equilibrium*

Proof: This is proved by showing that $\forall i \in N$, $\hat{v}_i^{ce} \geq 0$. The expression for $\hat{v}_i$ is quadratic in $\hat{p}_i$, and can be rewritten as $\hat{v}_i = a\hat{p}_j^2 + b\hat{p}_j + c$.

$$\hat{v}_i = A_{ii}\hat{p}_i^2 + ((1-\beta_i)\hat{p}_j A_{ij} + \beta_j p_j A_{ji})\hat{p}_i + (1-\beta_i)\hat{p}_j A_{ij}\bar{p}_i. \tag{30}$$

Because A_{ii} is negative, this expression represents a concave parabola (holding all variables other than $\hat{p}_i$ constant), which indicates that the vertex will be a maximum and will always be greater than or equal to the $\hat{v}_i$-axis intercept $(1-\beta_i)\hat{p}_j A_{ij}\bar{p}_i$. As the dynamics of $\hat{p}_i$ given in (17) maximize the above expression, if the value of the intercept is greater than or equal to 0, then the net effect of cooperation on firm i is non-negative.

Consider two cases: $sign(A_{ij}) = sign(\hat{p}_j)$ and $sign(A_{ij}) \neq sign(\hat{p}_j)$. In the first case, examining the signs of the factors shows that $(1-\beta_i)\hat{p}_j A_{ij}\bar{p}_i \geq 0$. Explicit limits on $\bar{p}_i$ and β_i, ensure that $\bar{p}_i$ and $(1-\beta_i)$ are non-negative. Furthermore, because $sign(A_{ij}) = sign(\hat{p}_j)$, the product of these two is also non-negative, and the entire term must be non-negative at equilibrium.

In the second case, $sign(A_{ij}) \neq sign(\hat{p}_j)$, which by lemma (5) implies β_i will be 1. As a result, $(1-\beta_i)\hat{p}_j A_{ij}\bar{p}_i = 0$. Therefore, the dynamics always result in all firms being at least as well off as they would be under non-cooperative dynamics. ■

From this, it also clearly follows that the producer surplus of the system is at least as great as it was at NCE. Furthermore, it can be shown that for any admissible linear demand function which has non-zero cross-price elasticity, each firm is strictly better off than it was at the NCE.

Lemma 6. *At equilibrium, if $\hat{p}_i = 0$, then $\beta_j = 1$.*

Proof: This follows directly from substitution into (29). ■

Lemma 7. *Given a linear demand function and the the dynamics in (17) and (24), at equilibrium, $(\hat{p}_i = \hat{p}_j = 0) \leftrightarrow (A_{ij} = A_{ji} = 0)$.*

Proof: First, we will prove that if at equilibrium $\hat{p}_i = \hat{p}_j = 0$, then $A_{ij} = A_{ji} = 0$. We proceed by contradiction, assuming that at equilibrium, $\hat{p}_i = \hat{p}_j = 0$ and $A_{ij}, A_{ji} \neq 0$. From lemma (6), $\beta_i = \beta_j = 1$. Substituting into (21),

$$\hat{p}_j = 0 = \frac{\bar{p}_i A_{ij}}{-2A_{ii}}, \tag{31}$$

which could only be true given an admissible linear demand if $A_{ij} = 0$. This is a contradiction. Therefore $(\hat{p}_i = \hat{p}_j = 0) \implies (A_{ij} = A_{ji} = 0)$.

The assertion that if $A_{ij} = A_{ji} = 0$, then $\hat{p}_i = \hat{p}_j = 0$, follows directly by substitution into (21). ■

This lemma shows that so long as cross-price elasticity of demand in the market is non-zero, cooperation will occur. We now proceed to show that this cooperation always results in both firms being strictly better off than they would be at the NCE.

Theorem 3. *Given an admissible linear demand function which demonstrates cross elasticity of demand, the equilibrium produced by the proposed system dynamics is such that both firms are better off than they would be at the NCE.*

Proof: The expression for $\hat{v}_i$ in (30) can be rewritten in vertex form

$$\begin{aligned} \hat{v}_i =& A_{ii}\left(\hat{p}_i - \frac{((1-\beta_i)A_{ij}\hat{p}_j + \beta_j A_{ji}p_j)}{-2a_{ii}}\right)^2 \\ &+ (1-\beta_i)\hat{p}_j A_{ij}\bar{p}_i - \frac{((1-\beta_i)\hat{p}_j A_{ij} + \beta_j p_j A_{ji})^2}{4A_{ii}}. \end{aligned} \tag{32}$$

At equilibrium then, $\hat{v}_i = (1-\beta_i)\hat{p}_j A_{ij}\bar{p}_i - \frac{((1-\beta_i)\hat{p}_j A_{ij} + \beta_j p_j A_{ji})^2}{4A_{ii}}$, and it can be shown that this value is always positive. Again, consider two cases: $sign(A_{ij}) = sign(\hat{p}_j)$ and $sign(A_{ij}) \neq sign(\hat{p}_j)$. In the first case, all terms are positive, so $\hat{v}_i > 0$. In the second case, $\beta_i = 1$, so again $\hat{v}_i > 0$. ■

In summary, the system demonstrates the desired properties at equilibrium. When cross-price elasticity of demand in the market is non-zero, cooperation induced through side payments makes both participants strictly better off. Importantly however, the dynamics do not guarantee a Pareto efficient equilibrium, as a later numerical example will show.

A pressing questions remains. The previous proofs and discussion assume two conditions: the existence of at least one reasonable equilibrium and a basin of attraction associated with the reasonable equilibrium which includes all reasonable initial conditions of the system. Solving for the equilibria of the system algebraically reveals that there in fact nine equilibria, but analysis of the properties of these equilibria proves difficult.

V. Numerical Example

Empirical evidence suggests that in fact there exists single reasonable equilibrium with a basin of attraction which includes all reasonable initial conditions. For a number of distinct demand functions, the system was simulated from six-hundred twenty-five initial conditions distributed evenly throughout the space of reasonable initial conditions. One such example is given below.

Consider a system defined by the following admissible linear demand function:

$$q(p) = \begin{bmatrix} -3 & 2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \begin{bmatrix} 6 \\ 6 \end{bmatrix}. \tag{33}$$

Profits and producer surplus of the system at equilibrium under non-cooperative, perfectly collusive and dynamically cooperative behavior are shown in the table below.

Dynamics	v_1 (\$)	v_2 (\$)	w (\$)
Non-Cooperative	5.98	4.58	10.56
Perfectly Collusive	8.00	4.00	12.00
Dynamic Coalition	6.43	4.90	11.33

Table I: In a two firm market selling substitute goods, dynamic coalition formation results in an increase in revenues for each player as well as an increase in producer welfare compared to the non-cooperative equilibrium

As expected, the total producer surplus under dynamic coalition dynamics is much closer than producer surplus under non-cooperative dynamics to the maximum producer surplus. Although perfect cooperation generates higher producer surplus, it is distributed in such a way that the second firm is better off not cooperating. Instead, the second firm would likely defect. Under dynamic coalition dynamics, however, both firms are better off.

The following table shows the prices and β-values of the system at equilibrium.

Dynamics	$\bar{p}_1$ (\$)	$\hat{p}_1$ (\$)	$\bar{p}_2$ (\$)	$\hat{p}_2$ (\$)	β_1	β_2
Non-Cooperative	1.41	0	1.24	0	0	0
Perfectly Collusive	1.41	0.59	1.24	0.76	0	0
Dynamic Coalition	1.41	0.15	1.24	0.27	0.48	0.41

Table II: In a two firm market selling substitute goods, prices and β values are reasonable at equilibrium. However, note that price does not converge to the optimal pricing policy.

Importantly, this equilibrium is reasonable, as both β values are between 0 and 1, and total price is positive. The cooperative equilibrium was the same for all 625 reasonable initial conditions, The image below shows the trajectory through the price space as the system evolves from four different initial conditions. Although the graph does not capture the equilibrium β values, these too converged to the same value regardless of the initial β and price values.

VI. Conclusion

This work introduced an explicit representation of a distribution mechanism in a game with transferable payoffs. It was shown that the explicit representation of the transfers in the form of side payments allows us to induce cooperation and resolve the question of distribution between cooperating

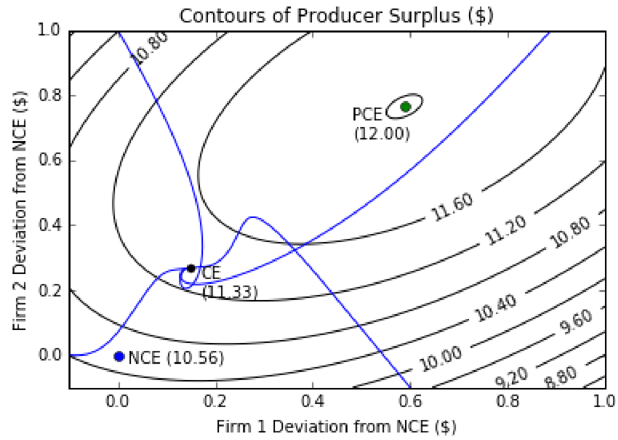

Figure 1: The convergence of the system from four different initial conditions to the CE demonstrates a basin of attraction which includes most if not all reasonable initial conditions.

players. The local stability and relative efficiency of the reasonable equilibria of the system are proved, and the probable existence and basin of attraction of such an equilibrium is shown in numerical examples.

Future work in this direction will focus on proving the existence and uniqueness of a reasonable equilibrium as well as investigation into the effect of the choice of β-dynamics on the final producer surplus and distribution between cooperating firms. Generalizing for a system with any number of players is also an important step to take towards designing a side payment mechanism which can dynamically distribute utility and induce cooperative coalitions.

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