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Necessary and Sufficient Conditions for Identifiability of Interconnected Subsystems

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Abstract— **Identifiability conditions refers to the information required, beyond input-output data, to *identify* the structure of the system. Since there are different ways to describe a system mathematically, there are different notions of structure associated with a single system. In this work we detail the identifiability conditions of interconnected subsystems, referred to as structured linear fractional transformations.**

The identifiability conditions of the structured linear fractional transformation are then compared to those of the dynamical structure function, another partial system representation, whose cost for identification was detailed in previous work [1]. Both representations appear to detail the same notions of structure of a system; however, this works demonstrates that the cost of identification of the structured linear fractional transformation is *always higher* than that of the dynamical structure function.

Understanding structural relationships in a system is critical for many reasons. The structure of a system can affect the development of control laws, analysis of security measures, or cost of implementation. There are many cases in which we may not know the exact structure of a system, even as a systems administrator, for example:

- in a biochemical reaction network, the complex interactions between various proteins may not be well understood, or
- as an attacker of an enemy system, some functionality of the system may be evident, but part of the intricate interconnection patterns may be obfuscated.

Since the only structural information that can be determined from input-output data is a system's transfer function (assuming sufficiency of excitation), identifiability conditions are important for understanding structures of unknown complex systems. The conditions state exactly what information is required a priori, beyond a system's input-output dynamics, in order to uniquely determine the structure of more detailed system representations.

Identifiability conditions for state space models were first studied by Glover and Willems, who demonstrated necessary and sufficient conditions for local identifiability. However, these conditions were only sufficient for global identifiability of a state space model [2]. These conditions were extended in work by Paré and Warnick to necessary and sufficient conditions for global identifiability of a state space model [3]. Identifiability conditions for the frequency domain representation of the dynamical structure function were developed in [1], while conditions of identifiability on the time domain representation were presented in [4].

This work is the first attempt to detail the identifiability conditions of the structured linear fractional transformation (commonly referred to as the interconnection of subsystems). The main goal of this work is to detail the high cost of identifying the structure linear fractional transformation in comparison to the dynamical structure function. The two representations are sometimes confused in the literature as the same representation [5], [6] even though they represent different notions of structure, which could cause issues when the reconstructed models are utilized for simulations, analysis, etc.

Section I details the background of each of the three system representations: the state space model, the dynamical structure function, and the structured linear fractional transformation. Then in Section II, we detail the necessary and sufficient conditions for identifiability of structured linear fractional transformations. Section III details the cost of identifiability. Next, Section IV explains the relationships between the identifiability conditions of the structured linear fractional transformation and dynamical structure function. Finally, conclusions and future work are presented in Section V.

I. Background

Consider a state space model of a system given by the equations:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{1}$$

The associated dynamical structure function of a state space model of the form (1) with C full row rank and $D = 0$ is:

$$Y(s) = Q(s)Y(s) + P(s)U(s) \tag{2}$$

where $Q(s)$ represents the manner in which measured states affect other measured states (potentially through hidden states) and $P(s)$ represents the manner in which inputs affect measured states (potentially through hidden states). The dynamical structure function was first developed in [7] and a comprehensive derivation of the dynamical structure function for a general set of state space equations was developed in [8].

Finally, the interconnection of subsystems, referred to in this work as structured linear fractional transformations, is

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given by the equations:

$$\begin{aligned} \begin{bmatrix} Y(s) \\ V(s) \end{bmatrix} &= N \begin{bmatrix} U(s) \\ W(s) \end{bmatrix} \\ W(s) &= S(s)V(s) \end{aligned} \tag{3}$$

where

$$N = \begin{bmatrix} 0 & I \\ L & K \end{bmatrix}$$

is a binary matrix representing the interconnection pattern of the subsystem and

$$S(s) = \begin{bmatrix} S_1(s) & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & S_q(s) \end{bmatrix}$$

is the block diagonal matrix whose entries correspond to the dynamics of the individual subsystems. Also, $W(s)$ is the aggregate of all outputs from subsystems and $V(s)$ is the aggregate of all inputs to subsystems, note that these can be external inputs or connections from other subsystems.

The original idea for describing an interconnection of subsystems (where each subsystem is a transfer function) as its own system representation was first presented by Sandberg and Murray, using a generalized version of (3) [9]. The definition used here was utilized in works by Yeung et al. to determine relationships between various system representations [10], [11].

Although there are many ways to write the structured linear fractional transformation, the definition in (3) was chosen because:

1) without loss of generality, every interconnection of subsystems can be put into this form, and
2) this definition details the interconnection of subsystems from the perspective that every output of each subsystem is measured.

The assumption that the output of each subsystem must be manifest may seem restrictive, but in the context that the structured linear fractional transformation is constructed by interconnecting existing systems, it is a reasonable assumption that the outputs of each of those systems is measured.

A. Information Cost Associated with Identifiability Conditions

The cost associated with the identifiability conditions of the state space model were shown to be roughly equivalent to ensuring full-state measurement, i.e. roughly n^2 parameters (entries of the matrices A, B, C, or D) must be known a priori in order to identify the unique state space realization that generated a specific set of input-output data [3]. A similar cost for the dynamical structure function shows that at least $p^2 - p$ entries of the transfer function matrices $Q(s)$ and $P(s)$ must be known a priori in order to identify the unique dynamical structure function that generated a specific set of input-output data [1].

Note that the information cost for state space model and the information cost for dynamical structure functions are not directly comparable since state space models are defined in the time-domain, while dynamical structure functions are defined in the frequency domain. Although a dynamical structure function representation was developed for the time domain in [4], the associated cost of identifiability is not immediately evident and is the subject of future work.

II. Identifiability Conditions

Consider the definition of the structured linear fractional transformation as given in (3). Solving for $Y(s)$ in terms of $U(s)$ yields:

$$Y(s) = (I - S(s)K)^{-1} S(s)LU(s), \tag{4}$$

assuming $(I - S(s)K)$ is invertible. That means that the closed loop transfer function of the structured linear fractional transformation is given by

$$G(s) = (I - S(s)K)^{-1} S(s)L. \tag{5}$$

The process of identification assumes that L, K, and $S(s)$ are unknown. Beginning with (5) and multiplying both sides by $(I - S(s)K)$ on the left and then collecting terms with unknown parameters on the right hand side yields:

$$G(s) = S(s)L + S(s)KG(s) \tag{6}$$

Taking the transpose of each side gives us:

$$G(s)^T = (S(s)L)^T + G(s)^T (S(s)K)^T \tag{7}$$

Restructuring the right hand side as a matrix multiplication yields

$$G(s)^T = \begin{bmatrix} I & G(s)^T \end{bmatrix} \begin{bmatrix} (S(s)L)^T \\ (S(s)K)^T \end{bmatrix} \tag{8}$$

which is of the form $Ax = b$, where A and b are known and x is unknown. Unfortunately, x is a matrix, so we can't solve $S(s)$, L, and K directly from (8). In order to put (8) in a form we can use, we apply the definition of the Kronecker product, which is

Definition 1. *Given $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$, the* Kronecker product *$A \otimes B$ is the $mp \times nq$ matrix:*

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix} \tag{9}$$

This means that (8) can be rewritten as:

$$\vec{g} = \begin{bmatrix} I \otimes I & I \otimes G(s)^T \end{bmatrix} \begin{bmatrix} \vec{sl} \\ \vec{sk} \end{bmatrix} \tag{10}$$

where $\vec{sl}$ is the vector taken by stacking the transpose of each row in $(S(s)L)^T$ vertically, similarly $\vec{sk}$ is the vector stack of $(S(s)K)^T$. Let $F = \begin{bmatrix} I \otimes I & I \otimes G(s)^T \end{bmatrix}$, then $F \in \mathbb{C}^{pm\times pm+p^2}$. Note that (10) is again in the form $Ax = b$, where A and b are known and x is the unknown. However, every entry of $\vec{sl}$ and $\vec{sk}$ are the sum of entries of $S(s)$ multiplied by entries of L and K respectively. Therefore,we need another step to separate the parameters being summed together.

Before we proceed, we first note that:

1) $S(s) \in \mathbb{C}^{p\times r}$, meaning there are rp unknowns in $S(s)$
2) $L \in \{0, 1\}^{r\times m}$, meaning there are rm unknowns in L
3) $K \in \{0, 1\}^{r\times p}$, meaning there are rp unknowns in K
4) Overall, there are $2rp + rm$ unknowns.

We will call the parameters in $S(s)$ the **dynamic** parameters, of which there are rp, and the parameters of L and K the **structural** parameters, of which there are $rp + rm$. The parameters of $S(s)$ and $\begin{bmatrix} L & K \end{bmatrix}$ multiplied together will then be **codependent** parameters. Since $S(s)$ is $p \times r$ and L is $r \times m$, there are $p \times m$ entries of $S(s)L$ each containing a sum of r entries, yielding pmr codependent parameters. Similarly, $S(s)K$ has p^2r codependent parameters. Therefore, the structured linear fractional transformation has $pmr+p^2r$ codependent parameters.

Given definitions for codependent parameters, we can apply the Kronecker product again to rewrite (10) as:

$$\vec{g} = \begin{bmatrix} I \otimes (I \otimes 1_{1\times r}) & I \otimes (G(s)^T \otimes 1_{1\times r}) \end{bmatrix} \begin{bmatrix} sl_1 \\ \vdots \\ sl_{pmr} \\ sk_1 \\ \vdots \\ sk_{p^2r} \end{bmatrix}$$

where sl_i is the i^{th} codependent parameter of $S(s)L$ and sk_j is the j^{th} codependent parameter of $S(s)K$. Let $H = \begin{bmatrix} I \otimes (I \otimes 1_{1\times r}) & I \otimes (G(s)^T \otimes 1_{1\times r}) \end{bmatrix}$, then $H \in \mathbb{C}^{pm\times pmr+p^2r}$.

Before we demonstrate the a priori information necessary and sufficient to identify the structured linear fractional transformation we need to show that we can identify the matrices L, K, and $S(s)$ from the codependent parameters. To do so, we first need the following definition:

Definition 2. *Dynamic and structural parameters are considered* active *if they are nonzero at least once with a corresponding parameter in a codependent parameter.*

Only active parameters can be decomposed into the entries of the matrices L, K, and $S(s)$, as demonstrated in the following Lemma.

Lemma 1. *The codependent parameters can be uniquely determined if and only if the active structural and dynamic parameters can be uniquely determined.*

Proof. Given the structural and dynamic parameters, the codependent parameters can be determined uniquely by multiplying the structural and dynamic parameters together. Specifically, given L, K and $S(s)$, we can compute $s_{ij}(s)l_{jh}$ and $s_{ij}(s)k_{jh}$ by multiplication of the corresponding entry.

Now, we look more closely at extracting the entries of L, K and $S(s)$ given the codependent parameters of the structured linear fractional transformation. Since L and K are boolean matrices, any non-zero codependent parameter can be decomposed into its structural and dynamic parameters. A structural or dynamic parameter r cannot be determined from the codependent parameters when every codependent parameter in which it appears is zero. If this is the case, then it is not an active parameter, which completes the proof. □

The final step necessary before detailing the identifiability conditions is the following Lemma:

Lemma 2. *Consider the $pmr + p^2r \times k$ transformation T such that $\hat{x} = Tz$, where z is an arbitrary vector of k transfer functions and*

$$\hat{x} = \begin{bmatrix} sl_1 & \dots & sl_{pmr} & sk_1 & \dots & sk_{p^2r} \end{bmatrix}^T$$

where A^T is the transpose of a matrix A. Let

$$\hat{M} = HT, \tag{11}$$

then $\hat{M}$ is injective if and only if $k \leq pm$ and $rank(\hat{M}) = k$.

Proof. This stems from the fact that $\hat{M}$ is a $pm \times k$ matrix, and $\hat{M}$ is injective if and only if it has full column rank, meaning $k \leq pm$ and $rank(\hat{M}) = k$. □

Now we are ready to proceed with the identifiability conditions of the structured linear fractional transformation:

Theorem 1. *Given a system characterized by the transfer function $G(s)$, the active parameters of its structured linear fractional transformation $(N, S(s))$ can be identified if and only if*

1) *$\hat{M}$, defined in (11), is injective, and*
2) *$\vec{g} \in \mathcal{R}(\hat{M})$*

Proof. The proof follows immediately from the observation that $\hat{M}$ is a mapping from unidentified model parameters to the system's transfer function, i.e. $\hat{M}z = \overleftarrow{g}$. □

Under the conditions from Theorem 1 one can solve for z given $G(s)$ and then use the reconstructed codependent parameters to determine the active structural and dynamic parameters in Lemma 1. We demonstrate the identifiability conditions of the structured linear fractional transformation through the following example:

Example 1. *Given the following transfer function of a system:*

$$G(s) = \begin{bmatrix} \frac{1}{s+1} \\ \frac{1}{(s+1)(s+2)} \end{bmatrix} \tag{12}$$

we attempt to find the structured linear fractional transformation of the system:

$$\begin{aligned} N &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ l_{11} & k_{11} & k_{12} \\ l_{21} & k_{21} & k_{22} \\ l_{31} & k_{31} & k_{32} \\ l_{41} & k_{41} & k_{42} \end{bmatrix} \\ S(s) &= \begin{bmatrix} s_{11}(s) & s_{12}(s) & s_{13}(s) & s_{14}(s) \\ s_{21}(s) & s_{22}(s) & s_{23}(s) & s_{24}(s) \end{bmatrix} \end{aligned} \tag{13}$$

Note that from the shape of matrices in (13) that $p = 2$, $m = 1$, and $r = 4$. We get the values for p and m directly from the shape of the transfer function, but it may not be immediately clear why $r = 4$. Given that $p = 2$ and we assume every output of a subsystem is measured, then there are at most $p = 2$ subsystems. Now, given that each subsystem can be an input to the other and we have $m = 1$ input, the number of inputs each subsystem can have is $p - 1 + m = 2$. This means we have two subsystems with two potential inputs each, yielding $r = p(p - 1 + m) = p^2 - p + pm = 4$.

From (13) the vector of unknown codependent parameters is

$$\hat{x} = \begin{bmatrix} s_{11}(s)l_{11} \\ s_{12}(s)l_{21} \\ s_{13}(s)l_{31} \\ s_{14}(s)l_{41} \\ s_{21}(s)l_{11} \\ s_{22}(s)l_{21} \\ s_{23}(s)l_{31} \\ s_{24}(s)l_{41} \\ s_{11}(s)k_{11} \\ s_{12}(s)k_{21} \\ s_{13}(s)k_{31} \\ s_{14}(s)k_{41} \\ s_{11}(s)k_{12} \\ s_{12}(s)k_{22} \\ s_{13}(s)k_{32} \\ s_{14}(s)k_{42} \\ s_{21}(s)k_{11} \\ s_{22}(s)k_{21} \\ s_{23}(s)k_{31} \\ s_{24}(s)k_{41} \\ s_{21}(s)k_{12} \\ s_{22}(s)k_{22} \\ s_{23}(s)k_{32} \\ s_{24}(s)k_{42} \end{bmatrix}. \tag{14}$$

From this we can derive the system of equations of the form $H\hat{x} = \overleftarrow{g}$ where

$$H = \begin{bmatrix} H_1 & H_2 & H_3 \end{bmatrix} \tag{15}$$

with

$$H_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$H_2 = \frac{1}{s+1}\begin{bmatrix} 1 & 1 & 1 & 1 & \frac{1}{s+2} & \frac{1}{s+2} & \frac{1}{s+2} & \frac{1}{s+2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$H_3 = \frac{1}{s+1}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & \frac{1}{s+2} & \frac{1}{s+2} & \frac{1}{s+2} & \frac{1}{s+2} \end{bmatrix}$$

Without additional information about the structure or dynamics of the system, we cannot reconstruct the structured linear fractional transformation.

Suppose that we know a priori that the boolean matrix L takes the form:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T \tag{16}$$

This knowledge reduces the number of unknowns in (14) from 24 codependent parameters to 18. This is still not enough for reconstruction.

Let us further assume that we also know that boolean matrix K takes the form:

$$K = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}^T \tag{17}$$

This knowledge further reduces the number of unknowns in from 18 codependent parameters to just 4. Surprisingly, this is still not enough a priori information for a unique reconstruction.

Finally, assume that we also know

$$\begin{aligned} s_{21}(s)l_{11} &= 0 \\ s_{13}(s)k_{31} &= 0 \end{aligned} \tag{18}$$

which leaves use with 2 unknowns in (14), which is sufficient for network reconstruction.

The vector of unknowns $\hat{x}$ can then be decomposed into the form $\hat{T}\hat{z}$ where $\hat{T}$ is a 24×2 binary matrix with a 1 only in the $(1, 1)$ and $(19, 2)$ entries. Then, we can get

$$\hat{z} = \begin{bmatrix} s_{11}l_{11} \\ s_{23}k_{31} \end{bmatrix} \tag{19}$$

Now, from Theorem 1 we can determine the equation $\hat{M}\hat{z} = \overleftarrow{g}$ where $\hat{M} = H\hat{T}$, given by

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} s_{11}l_{11} \\ s_{23}k_{31} \end{bmatrix} = \begin{bmatrix} \frac{1}{s+1} \\ \frac{1}{(s+1)(s+2)} \end{bmatrix} \tag{20}$$

In this case, $\hat{M}$ is full rank, so from Theorem 1 we know that the system is reconstructible. Since $\hat{z} = \hat{M}^{-1}\overleftarrow{g}$ we obtain the following structured linear fractional transformation:

$$\begin{aligned} N &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ S(s) &= \begin{bmatrix} \frac{1}{s+1} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{s+2} & 0 \end{bmatrix} \end{aligned} \tag{21}$$

which is the correct structured linear fractional transformation.

III. Information Cost of the Structured Linear Fractional Transformation

Example 1 demonstrates the main issue with reconstruction of the structured linear fractional transformation, that the size of each subsystem and number of subsystems are not known a priori. This means we need to reconstruct the system based on the worst case scenario that every subsystem is affected by every other subsystem and every external input. Unless we know r a priori, this means we need to determine $pmr + p^2r$ codependent parameters with an upper bound of $r \leq p^2 - p + pm$, i.e. an upper bound on the number of unknowns in a structured linear fractional transformation representation is

$$p^4 - p^3 + 2p^3m - p^2m + p^2m^2 \tag{22}$$

Lemma 3. *Let the amount of a priori information about the codependent parameters of the structured linear fractional transformation be denoted by the triple (x_1, x_2, x_3) where*

1) *x_1 be the number of nonzero codependent parameters, i.e. the structural and dynamic parameters are both nonzero,*
2) *x_2 the number of zero-valued structural parameters, and*
3) *x_3 the number of zero-valued dynamic parameters*

we also define x_4 to be the number of codependent parameters that include the overlap of elements that have both zero-valued structural and dynamic parameters known a priori.

Then the structured linear fractional transformation can be reconstructed if

$$x_1+px_2+(p+m)x_3-x_4 \geq p^4-p^3+2p^3m-p^2m+p^2m^2-pm \quad (23)$$

and the conditions of Theorem 1 hold.

Proof. From the definition of the codependent parameters we know that x_1 reduces one codependent parameter for each known codependent parameter. Moreover x_2 reduces the number of codependent parameters by p for each one that is known, while x_3 reduces the number of codependent parameters by $p+m$ for each one that is known. Finally, x_4 comes from the principle of inclusion and exclusion.

Since H is a $pm \times (pmr + p^2r)$ transfer function matrix and Theorem 1 requires $\hat{M}$ to be injective, we know we must reduce H to a $pm \times k$ transfer function matrix, where $k \leq pm$. This can only be achieved if the a priori information reduces the codependent parameters so that

$$pmr + p^2r - (x_1 + px_2 + (p+m)x_3 - x_4) \leq pm \quad (24)$$

which means

$$x_1 + px_2 + (p+m)x_3 - x_4 \geq pmr + p^2r - pm \quad (25)$$

Since we know that $r \leq p^2 - p + pm$, we use the upper bound to state that

$$x_1+px_2+(p+m)x_3-x_4 \geq p^4-p^3+2p^3m-p^2m+p^2m^2-pm \quad (26)$$

□

Applying the cost of reconstruction to Example 1, we see why so much information was required a priori in order to uniquely identify the correct structure. In the following example, we demonstrate how the information cost of Example 1 could have been determined.

Example 2. *Consider the transfer function in (12) where the unknown codependent parameters of the structured linear fractional transformation are given by the vector in (14), which has* 24 *unknown parameters. Example 1 had* $p = 2$, $m = 1$, *and* $r = 4$ *(as an upper bound). If we apply these values to the equation in (22) we get:*

$$2^4 - 2^3 + 2(2)^3(1) - 2^2(1) + 2^2(1^2) = 24 \quad (27)$$

Equation 26 from Lemma 3 states that

$$x_1+px_2+(p+m)x_3-x_4 \geq p^4-p^3+2p^3m-p^2m+p^2m^2-pm$$

i.e.

$$x_1 + 2x_2 + 3x_3 - x_4 \geq 22 \quad (28)$$

For Example 1, we know that from (16) and (17) we had $x_2 = 10$ *and from (18) we had* $x_1 = 2$, *with* $x_3 = 0$ *and* $x_4 = 0$. *Plugging these values into (28) yields:*

$$2 + 2(10) = 22 \geq 22$$

which, by Lemma 3, is sufficient to reconstruct the structured linear fractional transformation, as was done in (21).

IV. Comparison of Information Costs for Partial Structure Representations

Given these results, we will now focus on the comparison of the structured linear fractional transformation to the dynamical structure function. In this case, we first restrict ourselves to the case when each subsystem is a single input-single output transfer function (making it roughly equivalent to the dual of the dynamical structure function).

Imposing a restriction on the subsystems so that they are single-input, single-output transfer functions means that $S(s)$ is a diagonal matrix, which greatly reduces the number of unknown codependent parameters in the system meaning less a priori information is needed to reconstruct. Note that this restriction itself is equivalent to knowing information about the system a priori, namely that $S(s)$ is square and diagonal (i.e. $p^2 - p$ elements in $S(s)$ are zero).

Theorem 2. *Given that each subsystem in a structured linear fractional transformation* $(N, S(s))$ *is single-input, single-output, i.e. that* $S(s)$ *is diagonal, and letting* α *be the number of codependent parameters that are known a priori, then the conditions for reconstruction are that*

$$\alpha \geq p^2 - p \quad (29)$$

and the conditions of Theorem 1 hold.

Proof. Given that $S(s)$ is diagonal, we know $S(s)$ is a $p \times p$ transfer function matrix with p unknown dynamic parameters. Furthermore, L is a $p \times m$ boolean matrix with pm unknowns and K is a $p \times p$ matrix with $p^2 - p$ unknowns. The matrix $S(s)L$ yields pm unknown codependent parameters and the matrix $S(s)K$ yields $p^2 - p$ unknown codependent parameters. Therefore, there are $p^2 - p + pm$ unknown codependent parameters and in order to reduce $\hat{H} \in \mathbb{C}^{pm \times (p^2-p+pm)}$ so that $\hat{M}$ from Theorem 1 is injective, we require that the number of columns in $\hat{M}$ is reduced from $p^2 - p + pm$ to some value $k \leq pm$. This translates to

$$\alpha \geq p^2 - p + pm - pm \text{ or } \alpha \geq p^2 - p$$

Note that injectivity is not guaranteed in this case, but it is the minimum amount of information required to ensure the potential for injectivity of $\hat{M}$. □

Theorem 2 demonstrates that if we restrict the structured linear fractional transformation to conditions that make it appear like the dynamical structure function, then the information cost for both appear equivalent. Using a concrete example to demonstrate the change in cost when the restriction of single-input, single-output transfer functions are imposed.

Example 3. *Given the transfer function (12), we attempt to find the structured linear fractional transformation of the* *system:*

$$N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ l_{11} & k_{11} & k_{12} \\ l_{21} & k_{21} & k_{22} \end{bmatrix} \tag{30}$$

$$S(s) = \begin{bmatrix} s_{11}(s) & 0 \\ 0 & s_{22}(s) \end{bmatrix}$$

The associated codependent parameters are now given by

$$\hat{x} = \begin{bmatrix} s_{11}(s)l_{11} \\ s_{22}(s)l_{22} \\ s_{11}(s)k_{11} \\ s_{11}(s)k_{12} \\ s_{22}(s)k_{21} \\ s_{22}(s)k_{22} \end{bmatrix} \tag{31}$$

Another assumption of the dynamical structure function is that the $Q(s)$ matrix is hollow. This can be achieved in the structured linear fractional transformation by setting $k_{11} = 0$ and $k_{22} = 0$. This gives us the codependent parameters as

$$\hat{x} = \begin{bmatrix} s_{11}(s)l_{11} & s_{22}(s)l_{22} & s_{11}(s)k_{12} & s_{22}(s)k_{21} \end{bmatrix}^T \tag{32}$$

which gives us only 4 unknown parameters as opposed to the 24 from Example 1. This demonstrates the dramatic reduction in parameters brought about by the assumption that the structured linear fractional transformation is the dual of the dynamical structure function. Now, we can write $\hat{H}\hat{x} = \overleftarrow{g}$ as

$$\begin{bmatrix} 1 & 0 & \frac{1}{(s+1)} & 0 \\ 0 & 1 & 0 & \frac{1}{(s+1)(s+2)} \end{bmatrix} \begin{bmatrix} s_{11}(s)l_{11} \\ s_{22}(s)l_{22} \\ s_{11}(s)k_{12} \\ s_{22}(s)k_{21} \end{bmatrix} = \begin{bmatrix} \frac{1}{s+1} \\ \frac{1}{(s+1)(s+2)} \end{bmatrix} \tag{33}$$

Assume that we know $p^2 - p = 2^2 - 2 = 2$ codependent parameters a priori, as mentioned in Theorem 2. In particular, assume we know $s_{22}(s)l_{22} = 0$ and $s_{11}(s)k_{12} = 0$, then we can decompose $\hat{x}$ into $\hat{T}\hat{z}$ as follows:

$$\hat{T} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } \hat{z} = \begin{bmatrix} s_{11}(s)l_{11} \\ s_{22}(s)k_{21} \end{bmatrix} \tag{34}$$

Now, from Theorem 1 we can determine the equation $\hat{M}\hat{z} = \overleftarrow{g}$ where $\hat{M} = H\hat{T}$, given by

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} s_{11}(s)l_{11} \\ s_{22}(s)k_{21} \end{bmatrix} = \begin{bmatrix} \frac{1}{s+1} \\ \frac{1}{(s+1)(s+2)} \end{bmatrix} \tag{35}$$

which yields the structured linear fractional transformation

$$N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{36}$$

$$S(s) = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+2} \end{bmatrix}$$

V. Conclusion and Future Work

In this paper we demonstrated the information cost of identifying the structured linear fractional transformation of the form given in (3) from input-output data. We further showed that the cost of identifying a system's structured linear fractional transformation is always higher than the cost of identifying a dynamical structure function.

One potential avenue for future work would be to determine whether a more general class of structured linear fractional transformations would yield a more definitive relationship between the dynamical structure function and structured linear fractional transformation. A quick analysis demonstrates that a more general representation of the structure linear fractional transformation would yield a definition with more parameters. Increasing the number of unknown parameters would, in turn, increase the cost of identification of the structured linear fractional transformation. Increasing the information cost will still maintain the same basic relationship between the structured linear fractional transformation and dynamical structure function – although a more thorough analysis is required.

Another direction for future work is to develop a general framework of comparing information costs of any system representations, regardless of whether they are in the frequency or time-domain. Understanding how the costs of the dynamical structure function and structured linear fractional transformation compare to the cost of identifying the state space model could yield insights in how this may be accomplished.

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