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On the Well-Posedness of LTI Networks

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***Abstract*— We consider networks of linear, time-invariant systems defined over matrices of rational functions in a complex variable where each element of the matrix represents a link in the network. When these rational functions are proper, but not necessarily strictly proper, we demonstrate the necessary and sufficient conditions under which such a network configuration is well-posed. We include multiple examples of network configurations and their respective well-posedness conditions, including cases where two or more ill-posed network configurations can be interconnected to form a well-posed network.**

I. Introduction

When defining a model set to represent a class of dynamic systems, it can happen that some of the models in the set do not make physical sense. For example, if the outputs of a model are not uniquely determined by the inputs to the model, it makes sense to exclude this model from the model set, because this phenomena can not happen in the physical system that is being modeled. The notion of *well-posedness* is used to restrict a model set to models that make physical sense [1], [2], [3], [4], [5]. To quote from [1]: "well-posedness is essentially a modeling problem. It expresses that a mathematical model is, at least in principle, adequate as a description of a physical system ... Well-posedness thus imposes a regularity condition on feasible mathematical models for physical systems...In other words, since exact mathematical models would always be well posed, one thus requires this property to be preserved in the modeling."

In this work, we focus our attention on the well-posedness of networks of linear, time invariant (LTI) systems. We will use the *signal structure representation* of a network where nodes represent manifest variables, and links consist of single input, single output (SISO) dynamic systems.

Recent work (see, for example [6], [7]) has begun to explore the case where links in the signal structure may be causal, and not necessarily strictly causal. The interconnection of causal–but not necessarily strictly causal–systems may lead to ill-posed systems. As such, it is important to understand when a network containing causal links is well-posed and when it is not.

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In addition when given a representation of a system using n manifest variables, it is possible to remove variables from the model, and obtain a representation of the system using only m manifest variables (where $m < n$). We refer to this model as an *abstraction* of the original model. It can happen that even though the original model is well-posed, the abstraction may not be.

Given these observations, we pose the following questions: "what are the necessary and sufficient conditions to ensure well-posedness of a signal structure representation of a dynamic network?", and "how is the well-posedness of the signal structure representation related to the well-posedness of its abstractions?".

The paper is structured as follows. In Section III we define the signal structure representation of dynamic networks. In Section IV we present the definition of well-posedness and necessary and sufficient conditions for wellposedness of the signal structure representation of dynamic networks. In Section V we show that if an abstraction of the system is well-posed, then the signal structure representation is also well posed. We also show that the reverse implication is not true. In Section VI we present some examples. Finally, in Section VII we show that the well-posedness conditions can also be computed from the graph of the network.

II. Notation

Throughout this work, we will use the following notation:

- $\mathbb{P}(s)$: The set of all proper (not necessarily strictly proper) rational polynomials in s.
- $\mathbb{P}^{m\times n}(s)$: The set of all $m \times n$ matrices of rational polynomials in s.
- $\mathbb{H}^{n\times n}(s) \subset \mathbb{T}^{n\times n}$: The set of all $n \times n$ hollow (0's along the diagonal) matrices of rational polynomials in s.
- $g_{ij}(s)$ is the (i, j)'th entry of $G(s) \in \mathbb{T}^{m\times n}$. These entries are often called *links* or *modules*.
- $\Gamma(Q(s))$: The graph of $Q(s) = [q_{ij}(s)] \in \mathbb{H}^{n\times n}(s)$, where $y_1, \ldots, y_n$ enumerates the nodes of the graph and links are designated by the weighted adjacency matrix $M = [m_{ij}] \in \mathbb{R}^{n\times n}$ formed by the transpose of $Q(s)$ with $m_{ij} = 0$ if $q_{ji}(s) = 0$ and $m_{ij} = q_{ji}(\infty)$ otherwise.
- $(y_i, y_j, \ldots, y_k)$: Ordered tuples of nodes represent a *path* through $\Gamma(Q(s))$. Adjacent entries in this ordered tuple signify an entry in the $Q(s)$ of the signal structure representation. For example, path (y_i, y_j) is equivalent to $q_{ji}(s)$. A *cycle* is a path where the first and last nodes are equal. For example, (y_i, y_j, y_i) refers to the feedback loop from y_i to y_j and back to y_i again. The

length of the cycle is the number of *unique* nodes in the tuple, so the length of (y_i, y_j, y_i) is 2.

III. Signal Structures of Dynamic Networks

Consider the signal structure representation of a system $(Q(s), P(s))$ (as opposed to an interconnection of subsystems; see [8], [9], [10] for details and [11], [12], [13], [14] for applications) containing m inputs and p manifest variables, with

$$Y(s) = Q(s)Y(s) + P(s)U(s). \quad (1)$$

In the equation above, $U(s)$ and $Y(s)$ are the frequency domain representation of the manifest variables of the system. Additionally, $Q(s) = [q_{ij}(s)] \in \mathbb{H}^{p\times p}$ and $P(s) = [p_{ij}(s)] \in \mathbb{P}^{p\times m}$. Note that the transfer function representation of the system $G(s) \in \mathbb{P}^{p\times m}$ is given by

$$G(s) = (I - Q(s))^{-1} P(s). \quad (2)$$

Furthermore, let the state-space realization of each link $q_{ij}(s)$ be given by

$$\begin{aligned} \dot{x}_{q_{ij}}(t) &= A_{q_{ij}} x_{q_{ij}}(t) + B_{q_{ij}} y_j(t), \\ v_{q_{ij}}(t) &= C_{q_{ij}} x_{q_{ij}}(t) + d_{q_{ij}} y_j(t), \end{aligned} \quad (3)$$

with $A_{ij} \in \mathbb{R}^{n_{q_{ij}} \times n_{q_{ij}}}$, $B_{q_{ij}} \in \mathbb{R}^{n_{q_{ij}} \times 1}$, $C_{q_{ij}} \in \mathbb{R}^{1\times n_{q_{ij}}}$. and $d_{q_{ij}} \in \mathbb{R}$. Note also that

$$d_{q_{ij}} = q_{ij}(\infty). \quad (4)$$

Likewise, let the state-space realization of each link $p_{ij}(s)$ be given by

$$\begin{aligned} \dot{x}_{p_{ij}}(t) &= A_{p_{ij}} x_{p_{ij}}(t) + B_{p_{ij}} u_j(t), \\ v_{p_{ij}}(t) &= C_{p_{ij}} x_{p_{ij}}(t) + d_{p_{ij}} u_j(t), \end{aligned} \quad (5)$$

with $A_{ij} \in \mathbb{R}^{n_{p_{ij}} \times n_{p_{ij}}}$, $B_{p_{ij}} \in \mathbb{R}^{n_{p_{ij}} \times 1}$, $C_{p_{ij}} \in \mathbb{R}^{1\times n_{p_{ij}}}$. Note again that

$$d_{p_{ij}} = p_{ij}(\infty). \quad (6)$$

Define the set $\mathcal{X}_Q = \bigcup_{(i,j)\in Q(s),\, i\neq j} x_{q_{ij}}$ to be the set of all states in the realization of $Q(s)$. Likewise, define the set $\mathcal{X}_P = \bigcup_{(i,j)\in P(s)} x_{p_{ij}}$ to be the set of all states in the realization of $P(s)$.

Remark 1. *If the states in the state space realizations of these links are forced to be partitioned from system to system, then this network is a subsystem structure representation of the system. However, if state is allowed to be shared between links, then this network is known as a dynamical structure function or a linear dynamical network [9], [15], [16], [6], [17].*

In other words, it is possible–though not required–that $x_a(t) = x_b(t)$ for one or many choices of $a, b \in \mathcal{X}_Q \cup \mathcal{X}_P$ and for all t.

Suppose we have a system whose state-space representation contains n states. Let S_n be the signal structure $(Q_n(s), P_n(s))$ defined by measuring all n states, meaning $y_i = x_i$ for $i = 1, \ldots, n$. Then S_n contains the same amount of structural information as the state space representation.

Now suppose we wish to only measure $n-1$ of the states of the state space representation; or, in other words, we wish to abstract away one of the manifest variables, say y_i. We can then lift the network by solving for y_i in terms of y_j, $j \neq i$, and plugging in this solution to all instances of y_i in the network. The result will be a new signal structure, $(Q_{n-1}(s), P_{n-1}(s))$, which we will call S_{n-1}.

Thus S_{n-1} becomes an *abstraction* of the S_n; e.g. it is an abstraction of the state space representation. Furthermore, it is the unique abstraction corresponding to the lifting of y_i. Likewise, the state space representation–and hence S_n–is a *realization* of S_n, though this realization is not unique.

Suppose we repeat this process by taking S_{n-1} and abstracting away another manifest variable y_j. The result is a new signal structure S_{n-2} which is an abstraction of both S_{n-1} and S_n. Furthermore, S_{n-1} and S_n are both realizations of S_{n-2}. In otherwords, the abstraction and realization relationships are transitive (see Figure 1).

We can continue this process all the way to the point where we are measuring only one state, giving us the signal structure S_1. This signal structure has $Q(s)$ as a MISO transfer function; hence, it is contains the same level of structural information as the transfer function. By invoking the transitive nature of abstraction and realization, we can thus say that the transfer function S_1 is an abstraction of the state space representation S_n, and the state space representation S_n is a realization of the transfer function S_1, which is as expected.

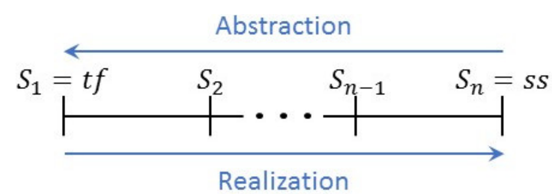


Fig. 1 The spectrum of network abstractions and realizations , from state space representations (far right) to transfer function representations (far left).

IV. On the Well-posedness of the Signal Structure Representation

In this work, we are treating the state space realization of the signal structure as the physical system. As such, we can adapt the definition of well-posedness given in [1] to the following:

Definition 1 (Well-Posedness of a Signal Structure)**.** *The signal structure of a system is* well-posed *if the following conditions are satisfied:*

(a) *The internal signals of all feedback loops in $(Q(s), P(s))$, namely $y_1, \ldots, y_p$, are uniquely defined for every choice of the realization state variables $x \in \mathcal{X}_Q \cup \mathcal{X}_P$ and external inputs $u_1, \ldots, u_m$.*

(b) *The internal signals of all feedback loops depend causally on the realization state variables and external inputs.*

(c) *The internal signals of all feedback loops depend continuously on the realization state variables and external inputs.*

(d) *Small changes in the model should not result in any feedback loop that does not satisfy the previous conditions.*

A signal structure that is not well-posed is called ill-posed.

Remark 2. *Many authors only use a subset of the conditions in Definition 1 to define well-posedness. For example, [1] and [18] use all four conditions, where [5] and [3] only use conditions (a-c), [2] uses only conditions (a-b), and [4] only uses condition (a).*

We are now prepared to explore the conditions of well-posedness on a signal structure with the following theorem:

Theorem 1 (Conditions of Well-Posedness on a Signal Structure)**.** *Consider the signal structure of an LTI system characterized by $(Q(s), P(s))$. This signal structure is well-posed if and only if $(I - Q(\infty))$ is invertible.*

Proof. We show each item in turn, noting that we drop the dependence on time from the notation for compactness.

Condition (a): We have that, for each $i = 1, \ldots, p$:

$$\begin{aligned} y_i &= \sum_{\substack{j=1 \\ j\neq i}}^{p} v_{q_{ij}} + \sum_{k=1}^{m} v_{p_{ik}} \\ &= \sum_{\substack{j=1 \\ j\neq i}}^{p} \left[C_{q_{ij}} x_{q_{ij}} + d_{q_{ij}} y_j\right] + \sum_{k=1}^{m} \left[C_{p_{ij}} x_{p_{ik}} + d_{p_{ik}} u_k\right]. \end{aligned} \tag{7}$$

Now define

$$y = \begin{bmatrix} y_1 & y_2 & \ldots & y_p \end{bmatrix}^T \tag{8}$$

$$u = \begin{bmatrix} u_1 & u_2 & \ldots & u_m \end{bmatrix}^T \tag{9}$$

$$C_Q(x_q) = \begin{bmatrix} 0 & C_{q_{12}} x_{q_{12}} & \cdots & C_{q_{1p}} x_{q_{1p}} \\ C_{q_{21}} x_{q_{21}} & 0 & \cdots & C_{q_{2p}} x_{q_{2p}} \\ \vdots & \vdots & \ddots & \vdots \\ C_{q_{p1}} x_{q_{p1}} & C_{q_{p2}} x_{q_{p2}} & \cdots & 0 \end{bmatrix}, \tag{10}$$

$$D_Q = \begin{bmatrix} 0 & d_{q_{12}} & \cdots & d_{q_{1p}} \\ d_{q_{21}} & 0 & \cdots & d_{q_{2p}} \\ \vdots & \vdots & \ddots & \vdots \\ d_{q_{p1}} & d_{q_{p2}} & \cdots & 0 \end{bmatrix}, \tag{11}$$

$$C_P(x_p) = \begin{bmatrix} C_{p_{11}} x_{p_{11}} & C_{p_{12}} x_{p_{12}} & \cdots & C_{p_{1m}} x_{q_{1m}} \\ C_{p_{21}} x_{p_{21}} & C_{p_{22}} x_{p_{22}} & \cdots & C_{p_{2m}} x_{p_{2m}} \\ \vdots & \vdots & \ddots & \vdots \\ C_{p_{p1}} x_{p_{p1}} & C_{p_{p2}} x_{p_{p2}} & \cdots & C_{p_{pm}} x_{p_{pm}} \end{bmatrix}, \tag{12}$$

$$D_P = \begin{bmatrix} d_{p_{11}} & d_{p_{12}} & \cdots & d_{p_{1m}} \\ d_{p_{21}} & d_{p_{22}} & \cdots & d_{p_{2m}} \\ \vdots & \vdots & \ddots & \vdots \\ d_{p_{p1}} & d_{p_{p2}} & \cdots & d_{p_{pm}} \end{bmatrix}. \tag{13}$$

Then Equation (7) can be written compactly as

$$y = C_Q(x_q) + D_Q y + C_P(x_p) + D_P u. \tag{14}$$

Rearranging, we get

$$(I - D_Q)y = C_Q(x_q) + C_P(x_p) + D_P u, \tag{15}$$

which will have a unique solution for y for any choice of u and x_q, x_p if and only if $(I - D_Q)$ is invertible. Plugging (4) into (11) reduces the condition to $(I - Q(\infty))$ needing to be invertible, as desired.

To complete the proof of this condition, note that the above condition holds for any choice of x_q and x_p, including the choice where a subset of these states are held equal to each other. Thus, this theorem holds for any realization of $(Q(s), P(s))$, including those that might have shared hidden state.

Condition (b): Equation (15) provides a state space representation of y with respect to x_q and x_p if and only if $I - D_Q = I - Q(\infty)$ is invertible (note that the equations for $\dot{x}$ are not given here but can be built by stacking the individual systems (3) and (5) and plugging in the solution for y in (15)), and since state space representations can only represent causal systems, y depends causally x_q, x_p, and u if and only if $I - Q(\infty)$ is invertible.

Condition (c): By Equation (15), y is continuous in x_q, x_p, and u if and only if $I - D_Q = I - Q(\infty)$ is invertible.

Condition (d): The only perturbation to $(Q(s), P(s))$ that will result in a change in the previous conditions is a perturbation that affects the static portion of $Q(s)$. In other words, this is a perturbation on $I - D_Q$. Suppose that $I - D_Q$ is full rank and thus satisfies conditions (a) through (c). Then, all singular values of $I - D_Q$ are strictly positive. As a result any perturbation on $I - D_Q$ must be at least as large as the smallest singular value of $I - D_Q$ in order to make it drop rank and fail to satisfy conditions (a) through (c). □

V. On the Well-Posedness of Network Abstractions

We can also say something concrete about the well-posedness of abstractions and realizations of a network.

Theorem 2. *Consider the signal structure S_n and an abstraction S_m, with $m < n$. If S_m is well posed, then S_n is well posed.*

Proof. We only show the proof to condition (a) on well-posedness here, conditions (b), (c), and (d) follow using the same logic as in Theorem 1.

All state space representations of S_n are characterized by Equation (15). For brevity, rewrite this equation as

$$(I - D_Q)y = R, \tag{16}$$

noting that the conditions for well-posedness of S_n are that $(I - D_Q)$ is invertible.

Now let y_2 be the $n - m$ manifest variables we are abstracting away from S_n to build S_m, and let y_1 be the remaining m manifest variables. Then we can permute and rewrite (16) as

$$\begin{bmatrix} I-D_{11} & -D_{12} \\ -D_{21} & I-D_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}. \tag{17}$$

Solving for y_2 in the bottom row of this equation gives

$$y_2 = (I-D_{22})^{-1}D_{21} + (I-D_{22})^{-1}R_2. \tag{18}$$

And plugging y_2 back in to the top row of (17) gives

$$((I-D_{11}) - D_{12}(I-D_{22})^{-1}D_{21})y_1 = R_1 + D_{12}(I-D_{22})^{-1}R_2, \tag{19}$$

which can be rewritten as

$$\begin{aligned} (I-\bar{D}_Q)y_1 &= \bar{R}, \\ \bar{D}_Q &= D_{11} + D_{12}(I-D_{22})^{-1}D_{21}, \\ \bar{R} &= R_1 + D_{12}(I-D_{22})^{-1}R_2. \end{aligned} \tag{20}$$

For this abstraction to exist, $\mathcal{D}_2 \triangleq I - D_{22}$ must be invertible. Also, S_m being well-posed implies, by Theorem 1, $\mathcal{D}_1 \triangleq I - \bar{D}_Q = (I-D_{11}) - D_{12}(I-D_{22})^{-1}D_{21}$ must be invertible. However, if both $\mathcal{D}_1$ and $\mathcal{D}_2$ are invertible, then $I - D_Q$ in Equation (17) is also invertible with inverse

$$\begin{bmatrix} \mathcal{D}_1^{-1} & \mathcal{D}_1^{-1}D_{12}\mathcal{D}_2^{-1} \\ \mathcal{D}_2^{-1}D_{21}\mathcal{D}_1^{-1} & \mathcal{D}_2^{-1} + \mathcal{D}_2^{-1}D_{21}\mathcal{D}_1^{-1}D_{12}\mathcal{D}_2^{-1} \end{bmatrix}$$

Since $(I - D_Q)$ invertible was the condition for S_n to be well-posed, we have that if S_m is well-posed then S_n is well-posed. □

Theorem 2 is sensible since an abstraction of a network system admits many realizations, not just the original network system. Well-posedness of an abstraction thus forces all of its realizations to be well-posed, including the original network system from which it was derived.

VI. Examples

We now apply Theorem 1 to various examples. The results of the first two are well known. The results of the remaining become less intuitive.

A. A Strictly Proper $Q(s)$

Let $Q(s)$ be strictly proper. Then $Q(\infty) = 0$. As such, $(I - Q(\infty)) = I$ and is always full rank. Therefore, any strictly proper $Q(s)$ will always be well-posed.

B. Two links in feedback

Consider an arbitrary $Q(s) \in \mathbb{H}^{2\times 2}$ as shown in Figure 2a. Then

$$I - Q(\infty) = \begin{bmatrix} 1 & -d_{12} \\ -d_{21} & 1 \end{bmatrix}.$$

In other words, this system contains a cycle at (y_1, y_2, y_1). This system is well-posed if and only if $\det(I - Q(\infty)) \neq 0$, which holds if and only if $1 - d_{12}d_{21} \neq 0$.

Note that this is precisely the well-known condition for well-posedness of two SISO systems connected in feedback.

C. A Ring

Now consider a $Q(s) \in \mathbb{H}^{3\times 3}$ where links are oriented in a ring such as that shown in Figure 2b. Without loss of generality, let

$$I - Q(\infty) = \begin{bmatrix} 1 & 0 & -d_{13} \\ -d_{21} & 1 & 0 \\ 0 & -d_{32} & 1 \end{bmatrix}.$$

Once again, this system is well-posed if and only if $\det(I - Q(\infty)) \neq 0$ which is true if and only if $1 - d_{13}d_{21}d_{32} \neq 0$.

D. The Adjoining of Two Feedback Loops

Let

$$I - Q(\infty) = \begin{bmatrix} 1 & -d_{12} & 0 \\ -d_{21} & 1 & -d_{23} \\ 0 & -d_{32} & 1 \end{bmatrix}.$$

The condition of well-posedness of this system is that $1 - d_{12}d_{21} - d_{23}d_{32} \neq 0$.

Notice from Figure 2c that this example contains two cycles of the type described in Section VI-B, namely (y_1, y_2, y_1) and (y_2, y_3, y_2). However, the condition $(1 - d_{12}d_{21} - d_{23}d_{32} \neq 0)$ is different than the union of conditions on the individual loops ($1 - d_{12}d_{21} \neq 0$ *and* $1 - d_{23}d_{32} \neq 0$). As such, it is possible for each of the two cycles to be ill-posed, but the overall system adjoining these two cycles at output 2 is actually well-posed.

To illustrate this, let

$$I - Q(\infty) = \begin{bmatrix} 1 & -.5 & 0 \\ -2 & 1 & -2 \\ 0 & -.5 & 1 \end{bmatrix}. \tag{21}$$

Notice that the cycle (y_1, y_2, y_1) is ill-posed since $1 - d_{12}d_{21} = 1 - (.5)(2) = 0$. Likewise, the cycle (y_2, y_3, y_2) is ill-posed since $1 - d_{23}d_{32} = 1 - (2)(.5) = 0$. However, the overall system is actually well-posed since $1 - d_{12}d_{21} - d_{23}d_{32} = 1 - (.5)(2) - (2)(.5) = -1 \neq 0$.

It is also possible to adjoin two well-posed cycles to make the overall system ill-posed. To illustrate this, let

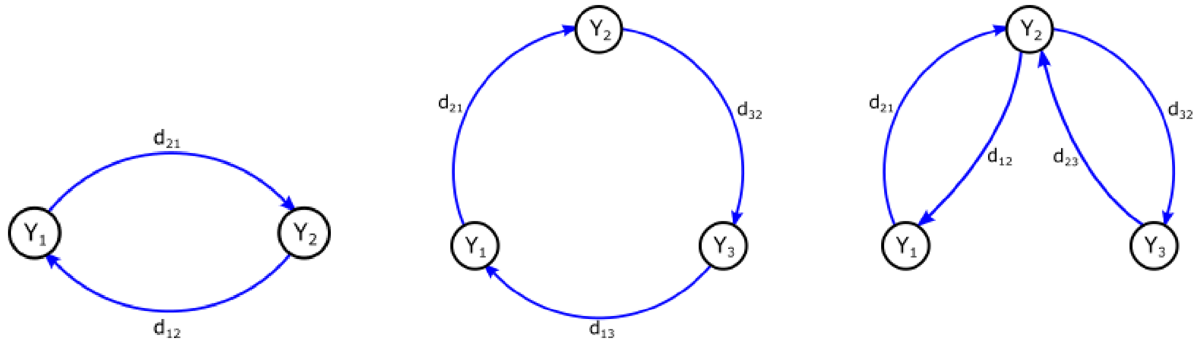
$$I - Q(\infty) = \begin{bmatrix} 1 & -.5 & 0 \\ -1 & 1 & -1 \\ 0 & -.5 & 1 \end{bmatrix}. \tag{22}$$

Notice that the cycle (y_1, y_2, y_1) is well-posed since $1 - d_{12}d_{21} = 1 - (.5)(1) = .5 \neq 0$. Likewise, the cycle (y_2, y_3, y_2) is well-posed since $1 - d_{23}d_{32} = 1 - (1)(.5) = .5 \neq 0$. However, the overall system is actually ill-posed since $1 - d_{12}d_{21} - d_{23}d_{32} = 1 - (.5)(1) - (1)(.5) = 0$.

E. Well-posedness of Abstractions

Suppose that Equation (16) is given by:

$$\begin{bmatrix} 1 & d_{12} & 0 & d_{14} \\ d_{21} & 1 & 0 & 0 \\ 0 & d_{32} & 1 & d_{34} \\ 0 & 0 & d_{43} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} \tag{23}$$



(a) A network with two links in feedback, which is well-posed if and only if $1 - d_{12}d_{21} \neq 0$.

(b) A network with three links oriented in a ring, which is well-posed if and only if $1 - d_{13}d_{21}d_{32} \neq 0$.

(c) A network four links, which is well-posed if and only if $1 - d_{12}d_{21} - d_{23}d_{32} \neq 0$.

Fig. 2 Example networks discussed in Sections VI-B, VI-C, and VI-D respectively.

The expression for $\det(I - Q(\infty))$ in this case is:

$$\det(I - Q(\infty)) = 1 - d_{14}d_{43}d_{32}d_{21} - d_{12}d_{21} - d_{43}d_{34} + d_{12}d_{21}d_{34}d_{43}. \quad (24)$$

Therefore, by Theorem 1, this signal representation is well-posed as long as (24) does not equal zero. However, suppose that we want to abstract the manifest variables y_3 and y_4 away from the representation; i.e. we want a network representation only in terms of y_1 and y_2. This can be done by eliminating y_3 and y_4 from (23); i.e. express y_1 and y_2 only in terms of y_1 and y_2, and r_i, $i = 1, 2, 3, 4$. The resulting equation is:

$$\left(\begin{bmatrix} 1 & d_{12} \\ d_{21} & 1 \end{bmatrix} + \begin{bmatrix} 0 & d_{14} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & d_{34} \\ d_{43} & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & d_{32} \\ 0 & 0 \end{bmatrix}\right) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} - \begin{bmatrix} 0 & d_{14} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & d_{34} \\ d_{43} & 1 \end{bmatrix}^{-1} \begin{bmatrix} r_3 \\ r_4 \end{bmatrix} \quad (25)$$

which can be rewritten as

$$\begin{bmatrix} \breve{r}_1 \\ \breve{r}_2 \end{bmatrix} = \begin{bmatrix} 1 & d_{12} + \frac{d_{14}d_{32}d_{43}}{1 - d_{34}d_{43}} \\ d_{21} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = Q_2(\infty) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

For this abstraction to be well-posed, we need that $\det(I - Q_2(\infty)) \neq 0$. We have that

$$\det(I - Q_2(\infty)) = \frac{\det(I - Q(\infty))}{1 - d_{43}d_{34}}.$$

Thus $1 - d_{43}d_{34} \neq 0$ and $\det(I - Q(\infty)) \neq 0$, the latter condition guaranteeing that the original signal representation is well-posed as well, as predicted by Theorem 2.

VII. Computing the Well-Posedness Conditions from the Graph Structure

Notice from the previous examples that all well-posedness conditions have a particular form. We can generalize this form to create a method for computing the determinant of $I - Q(\infty)$ from the weighted graph created by the adjacency matrix formed by $Q(\infty)$.

Let

$$D = I - Q(\infty) = \begin{bmatrix} 1 & d_{12} & \cdots & d_{1p} \\ d_{21} & 1 & \cdots & d_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ d_{p1} & d_{p2} & \cdots & 1 \end{bmatrix} \quad (26)$$

and let $\Gamma(D)$ be the graph of this weighted adjacency, noting that we include the self loops of weight 1 that are lacking from $Q(\infty)$ (recall again that we use the transpose of the standard definition for adjacency matrices to define our graphs).

The Leibniz formula for determinants gives that

$$\det(D) = \sum_{\sigma \in S_p} sign(\sigma) \prod_{i=1}^{p} d_{\sigma(i),i}, \quad (27)$$

where S_p is the set of all permutations of $\{1, \ldots, p\}$ and $sign(\sigma)$ is the parity of permutation σ. In other words, the determinant of D contains products of four terms of D, where the product is multiplied by ± 1 and where no two terms in the product can share the same row or the same column, and the determinant itself is the sum of all such products.

These permutations used in computing the determinant have a strong tie to cycles in $\Gamma(D)$. In particular, each $\prod_{i=1}^{p} d_{\sigma(i),i}$ is the product of the weights on disjoint cycles, and all possible combination of disjoint cycles are considered. Furthermore $sign(\sigma)$ is -1 if and only if the permutation contains an odd number of even-length disjoint cycles, and $+1$ otherwise. Note that it does not matter that we are using the transpose of the typical definition of graphs, as the determinant of a matrix is equal to the determinant of its transpose.

An immediate consequence of this is that $\sigma = \{1, \ldots, p\}$ is a permutation of $\{1, \ldots, p\}$, corresponding to $\prod_{i=1}^{p} d_{\sigma(i),i} = \prod_{i=1}^{p} d_{ii} = 1$ (since every diagonal entry of D is 1). Furthermore, since σ can be converted into $\{1, \ldots, p\}$ with 0 transpositions, we have $sign(\sigma) = +1$. Hence 1 is always a term in the determinant. This term corresponds to the case where we consider the p disjoint self loops on the graph structure.

We illustrate the computation of the well-posedness conditions of a signal structure from the graph of the network with an example. Let

$$D = I - Q(\infty) = \begin{bmatrix} 1 & d_{12} & 0 & 0 & 0 \\ 0 & 1 & d_{23} & 0 & 0 \\ d_{31} & 0 & 1 & 0 & 0 \\ 0 & d_{42} & 0 & 1 & d_{45} \\ 0 & 0 & d_{53} & d_{54} & 1 \end{bmatrix}. \quad (28)$$

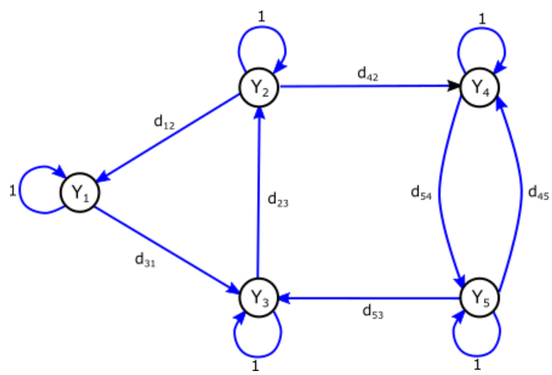


Fig. 3 The graph of Equation (28).

The graph of D shown in Figure 3 contains the following possible sets of disjoint cycles with their respective products and signs:

- All self loops. The product is 1 and the sign is +1 since there are no cycles of even length.
- $(y_1, y_3, y_2, y_1), (y_4, y_4), (y_5, y_5)$. The product is $d_{12}d_{31}d_{23}(1)(1) = d_{12}d_{31}d_{23}$, and the sign is $+1$ since there are no cycles of even length.
- $(y_4, y_5, y_4), (y_1, y_1), (y_2, y_2), (y_3, y_3)$. The product is $d_{45}d_{54}(1)(1)(1) = d_{45}d_{54}$, and the sign is -1 since there is one cycle of even length.
- $(y_1, y_3, y_2, y_1), (y_4, y_5, y_4)$. The product is $d_{12}d_{31}d_{23}d_{45}d_{54}$, and the sign is -1 since there is one cycle of even length.

Therefore, $det(D) = 1 + d_{12}d_{31}d_{23} - d_{45}d_{54} - d_{12}d_{31}d_{23}d_{45}d_{54}$.

VIII. Related Work and Contributions

This work is built off the definitions of well-posedness originally presented in [1]. In [18], a sufficient condition for the well-posedness of interconnected networks was given. A condition for well-posedness of signal structures was also considered in [19], which proposes that the signal structure is well-posed if every principal minor of $I - Q(\infty)$ is nonzero. Again, this condition is only a sufficient condition, and is not necessary. See Equation (21), which demonstrates well-posed networks where there exist some principal minors of $I - Q(\infty)$ that are zero.

In [6], it was properly stated that well-posedness implies that $I - Q(\infty)$ has a proper inverse, but refers to [19] for the proof, which, as was stated previously, only contains a proof of sufficiency.

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X. Conclusions

In conclusion, we have presented and provided proof for the necessary and sufficient conditions of signal structure, which are that the signal structure is well posed if and only if $I - Q(\infty)$ is invertible.

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