

Abstractions and Realizations of Dynamic Networks

Nathan Woodbury[†] and Sean Warnick[†]

Abstract—This paper establishes the importance of abstractions and realizations of dynamic networks in characterizing the structure and dynamics of systems. Abstractions and realizations generate dynamically equivalent representations of systems with varying degrees of structural detail. We show that dynamic networks exist that contain the same level of detail as state space models, that other dynamic networks exist that contain the same level of detail as transfer functions, and that still other dynamic networks exist that are simultaneously abstractions of state space models and realizations of transfer functions; thus containing intermediate levels of structural detail.

I. INTRODUCTION

We can represent the dynamics of an LTI system in many ways, such as through a state space model or a transfer function. For applications where knowledge of both the dynamic behavior and the computational structure of the system are required, dynamic networks have increasingly become the representation of choice [1]–[5]. We begin by introducing dynamic networks as represented by the dynamical structure function (DSF).

A. Background: Dynamic Networks

Let W be the set of N signals, w_1, \dots, w_N , that are manifest (or visible) in some dynamic system. Dynamic networks are representations that are characterized by the dynamic dependencies among these signals. Causal dependencies among subsets of signals impose directions on these dependencies. As such, we can draw a graph (or hyper-graph if we are considering some classes of nonlinear systems) of the system with the manifest variables as nodes and the computational or causal dependencies forming the edges. We call this graph the *signal structure* of the system. This paper is concerned primarily with linear, time-invariant (LTI) systems, though many of the concepts discussed here remain fundamental in a non-linear or a non-causal setting.

For LTI systems, we can characterize any signal structure with an adjacency matrix \bar{Q} , where each entry \bar{Q}_{ij} is a single-input, single-output (SISO) system (which can be defined either in continuous time or in discrete time), and where the dependency of any signal w_i on itself is 0, meaning each $\bar{Q}_{ii} = 0$. Then the behavior of the system can be characterized by $(I - \bar{Q})w = 0$, or equivalently $w = \bar{Q}w$. For simplicity, we will assume that we are representing continuous-time systems in the frequency domain, though

the analysis in this paper applies as well to discrete-time and time-domain representations as well.

In the graph of \bar{Q} , there exists a (possibly empty) subset of manifest variables that are independent of all other manifest variables. These correspond to the nodes in the signal structure graph with in-degree equal to zero, and likewise, correspond to rows within \bar{Q} that are equal to zero. We can label this subset of manifest variables as *inputs*¹ U , and label the remaining manifest variables as outputs Y . By permuting and partitioning the behavior \bar{Q} , we get the following structured representation of the behavior of the system:

$$\begin{bmatrix} Y \\ U \end{bmatrix} = \begin{bmatrix} Q & P \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Y \\ U \end{bmatrix} \implies Y = QY + PU. \quad (1)$$

The pair (Q, P) is known as the *dynamical structure function* (DSF) representation of this system, and is the primary focus of this work.

We can re-arrange the DSF such that

$$Y = [(I - Q)^{-1}P]U \triangleq GU, \quad (2)$$

where G is the transfer function characterizing the input-output behavior of this system. Thus we see that the DSF is a left factorization of the input-output map. We also see that the ability to compute the inverse of $(I - Q)$ is important in characterizing DSFs, and in fact, the ability to compute this inverse is intimately tied to the notion of *well-posedness* [6], [7].

B. Abstractions and Realizations of Dynamic Networks

As we detail in Section III, an abstraction of some DSF (Q, P) is another dynamic network (such as a DSF) with equivalent input-output behavior but with *less* structural detail. Similarly, a realization of (Q, P) is another dynamic network, again with equivalent input-output behavior but with *more* structural detail. For network reconstruction, this implies that an abstraction is cheaper to reconstruct than the original DSF. Thus, abstractions provide two critical benefits which motivate this work:

- Abstractions allow us to reduce the cost of reconstructing networks regarding the amount of *a priori structural information* needed (see Section II-A).
- Abstractions represent a system at varying levels of detail. In studies of robustness and vulnerability of a networked system (see, for instance, [8], [9]), this allows us to study both the vulnerability at an outsider's

[†]Nathan Woodbury (nathanswoodbury@gmail.com) and Sean Warnick (sean.warnick@gmail.com) are with the Information and Decision Algorithms Laboratories, Department of Computer Science, Brigham Young University, Provo, UT 84602, USA.

¹We make no distinction between controlled, uncontrolled, deterministic, stochastic, measured, or unmeasured inputs.

level of information (abstraction with less structural detail) or an insider’s level of detail (realization with more structural detail).

In this work, we discuss two different types of abstractions, the immersion and the stacked immersion.

The *immersion* (which has at times been simply called the *abstraction*) has been previously explored in many ways. The term “immersion” was introduced in [10] in reference to the output elimination procedure described in Section III-A. The works [10]–[12] leverage the reduced amount of identifiability conditions required for reconstructing an immersion to identify specific links in the network as opposed to the network as a whole. In [13]–[16], the DSF itself is described as simultaneously an immersion of a state space model and a realization of a transfer function. This idea is formalized in [17], which discusses the identifiability of a state space model from a DSF and provides necessary and sufficient conditions under which an immersion creates hidden states in the network that are shared among the edges. The works [6], [7] give conditions on when the immersion is proper and well-posed.

The stacked immersion is a new type of abstraction introduced in this work, which we construct using the immersion as a foundation. A key idea of this paper is that the stacked abstraction can be used to form a spectrum of models ranging from a state space model to a transfer function as ordered by a measure of structural complexity known as the *identifiability index*.

C. Contributions

This paper centers around DSFs as a representation of LTI systems as compared to other representations such as state space models or transfer functions. We do this by:

- (i) Defining an immersion and showing (for the first time²) that this abstraction preserves the dynamics, independence pattern, and hollowness of the original network (Section III-A and Theorem 1).
- (ii) Introducing the stacked immersion and the multi-DSF (Section III-B).
- (iii) Showing that a DSF (and hence a multi-DSF) exists that is informationally equivalent to a state space model and that another exists that is equivalent to the transfer function (Section IV).
- (iv) Showing that multi-DSFs exist that are informationally intermediate between a state space model and the transfer function (Section IV)³.

²Theorem 1 is similar to Propositions 1 and 2 in [10] in that it shows that the immersion procedure meets the immersion definition and preserves hollowness. However, the novelty of this work is a more complete definition of the immersion, requiring dynamic equivalence and the independence pattern to be preserved as well. We show that the immersion procedure does indeed preserve these properties.

³This is related to the results in [17] which uses the generic McMillan degree to compare the information contained in a state space model to that contained in a DSF. Here, we use the necessary conditions (as measured by the identifiability index, see Section II-A) instead of the generic McMillan degree as this allows us to cleanly create a partial ordering of all dynamically equivalent DSFs between and including the state space model and the transfer function.

II. BACKGROUND

Much of this work centers around functions of the form $g(s) = \frac{n(s)}{d(s)}$, where $n(s)$ and $d(s)$ are polynomials in $s \in \mathbb{C}$. If the degree of $n(s)$ is no greater than the degree of $d(s)$, we say that $g(s)$ is *proper*. If the degree of $n(s)$ is strictly less than the degree of $d(s)$, then we say that $g(s)$ is *strictly proper*. We say that matrices of rational functions are (strictly) proper if every entry in the matrix is (strictly) proper. We define $RP^{m \times n}$ as the set of all $m \times n$ matrices of proper (and not necessarily strictly proper) rational functions.

We say that $G \in RP^{p \times p}$ is *invertible almost everywhere* if it is singular only for a finite choice of values $s \in \mathbb{C}$. For brevity, from this point forward, when we say that G is invertible, we mean that it is invertible almost everywhere.

Any arbitrary DSF (Q, P) , with $Q \in RP^{p \times p}$ and $P \in RP^{p \times m}$ (note that neither are required to be strictly proper), is *well-posed* (i.e. algebraically consistent, or simulatable) if and only if $(I - Q)$ has a proper inverse. We assume that every DSF and every immersion discussed in this work are both proper and well-posed (i.e., we will implicitly assume that every inverse exists, is unique, and is proper). A sufficient condition for any DSF and all of its immersion to be well-posed is that every principle submatrix of $(I - Q)$ has a proper inverse. See [6], [7] for a complete discussion on these points.

A. Network Reconstruction

Let $G \in RP^{p \times m}$ be a transfer function that is either given or identified from data. Network reconstruction is the process of leveraging structural information (which will be defined below) to find the *unique* DSF (Q, P) such that $Q \in RP^{p \times p}$, $P \in RP^{p \times n}$, and $G = (I - Q)^{-1}P$. Note that none of these matrices need be strictly proper, and the network reconstruction methodology presented below works even if the operators are improper or even functions over s that are not rational functions. We will focus our discussion here, however, on the case where Q and P are matrices of rational functions.

As shown in [18]–[20], we can recover Q and P by writing

$$G = (I - Q)^{-1}P = [Q \ P] \begin{bmatrix} G \\ I_p \end{bmatrix} \implies \quad (3a)$$

$$G' = [G' \ I_p] \begin{bmatrix} Q' \\ P' \end{bmatrix} \implies \quad (3b)$$

$$\vec{g} = L\vec{\theta}, \quad (3c)$$

where $\vec{g} \in RP^{pm}$ is a vectorization of the pm known entries in G' , $\vec{\theta} \in RP^{p^2+pm}$ is a vectorization of the $p^2 + pm$ unknown entries in $[Q \ P]'$, and $L \in RP^{pm \times (p^2+pm)}$ is constructed to preserve the transformation $[G' \ I_p]$ between the new vectorized spaces.

Note every entry in L and \vec{g} is known, so the network reconstruction problem reduces to finding a unique $\vec{\theta}$. However, since $p^2 + pm > pm$, this problem is ill-posed. Hence, we must reduce the dimensions of the columns such that the number of columns is less than or equal to pm . Since, by definition, Q is hollow ($Q_{ii} = 0$), we can remove every

column of L (and row in $\vec{\theta}$) corresponding to each Q_{ii} , which reduces the number of columns to $p^2 - p + pm$. However, we must somehow eliminate another $p^2 - p$ columns. We can do so by leveraging *a priori structural information* about the DSF, which may take the following form

- Knowledge that an entry in P or Q is zero.
- Knowledge that an entry in P or Q is a known linear combination of other unknown entries in P or Q .

If we know *at least* $p^2 - p$ entries in P and Q according to the above criteria, then we can recover Q and P uniquely. We refer to this number $p^2 - p$ as the *identifiability index*, which is the measure of the amount of the minimum amount of a priori structural information that is necessary to reconstruct (Q, P)

B. The Signal Structure

Let (Q, P) be a DSF defining the relationship $Y = QY + PU$ with $Q \in RP^{p \times p}$, $P \in RP^{p \times m}$, $Y = [y_1 \ \dots \ y_p]'$, and $U = [u_1 \ \dots \ u_m]'$. Associated with this DSF is a weighted directed graph called the *signal structure* $\Gamma(Q, P) = (V, E, W)$, where the vertices are defined by the set

$$V = \{y_1, \dots, y_p\} \cup \{u_1, \dots, u_m\}, \quad (4)$$

and the edges are defined by the set

$$E = \{(y_i, y_j) \mid i, j = 1, \dots, p\} \cup \{(u_i, y_j) \mid i = 1, \dots, m, j = 1, \dots, p\}. \quad (5)$$

Furthermore, W assigns weights to the edges by defining the relation $W : E \times E \rightarrow RP$ (note that the edge weights are in RP rather than \mathbb{R} as is common in graph theory) such that

$$W(y_i, y_j) = Q_{ji}, \quad W(u_i, y_j) = P_{ji}. \quad (6)$$

In other words, the edges are defined by the Boolean structure of Q and P and the weights are defined by the entries in Q and P . We will also discuss $\Gamma(Q)$, which is a sub-graph of $\Gamma(Q, P)$ consisting only of the nodes $\{y_1, \dots, y_p\}$ and the edges connecting those nodes.

Edges in the signal structure are essential to the interpretation of the DSF. If there exists a (non-zero) edge from y_i to y_j , this means that there is a component of signal y_j that is computed directly from y_i and independent of all other manifest signals y_k and u_l in the network (and likewise for an edge from u_i to y_j). Furthermore, since Q and P are matrices of proper rational functions, this computation is causal. However, if the edge from y_i to y_j is zero, this means that either (i) there is no direct computation of y_j from y_i independent from all other manifest signals, or (ii) that the direct computation has been canceled exactly across the latent variables in the network. Thus, the signal structure represents the direct and causal dependence among variables manifest in the network. We call these direct causal dependencies and independencies the *independence pattern* of the network.

We can also import and adapt existing graph-theoretic terms and concepts. Of greatest import to this work is the

notion of a *walk*, with the notion of *net effect* defined over the walk.

Definition 1 (Walk) Let (Q, P) be a DSF with corresponding $(V, E, W) = \Gamma(Q, P)$. A *walk* from $y_i \in V$ (or $u_i \in V$) to $y_j \in V$ is a sequence of edges $[(y_i, y_{k_1}) \in E, \dots, (y_{k_m}, y_j) \in E]$ such that the first edge starts at y_i and the last edge terminates at y_j . In a walk, vertices and edges are both allowed to repeat in the sequence. \square

Definition 2 (Net Effect) We say that the *net effect* of some walk is the product of the weights of all edges in the walk⁴. \square

III. ABSTRACTIONS

We now discuss the two types of abstractions that form the focus of this paper, the immersion and the stacked immersion.

A. Immersions

Suppose that we wish to stop observing or modeling certain output signals in some network. In other words, suppose we wish to reduce the set of variables that are manifest in the system. Since we define links in the DSF as the interaction between a source and destination manifest variable, independent from other manifest variables, the DSF must change in order to reflect our new manifest set without discarding the internal dynamics of the system. The process of removing a set of nodes (output signals) is called *node abstraction*, but that can leave a representation of a dynamic network with a non-hollow Q matrix. The subsequent *hollow abstraction* then removes self-loops in the network, thereby yielding an admissible DSF. We compose these two processes as follows:

Definition 3 (Immersion) Let (Q, P) define the relationship $Y = QY + PU$. Let S index some subset of the outputs Y with \bar{S} indexing the remaining outputs. An *immersion* of some DSF (Q, P) with respect to S is a DSF (Q_S, P_S) such that the following properties are preserved from (Q, P) :

(P1) *Dynamics*: If $Y_S = Q_S Y_S + P_S U_S$, then

$$\begin{bmatrix} Y_S \\ Y_{\bar{S}} \end{bmatrix} = (I - Q)^{-1} P U \implies Y_S = (I - Q_S)^{-1} P_S U.$$

In other words, if S indexes the first $|S|$ outputs in Y , the transfer function $(I - Q_S)^{-1} P_S$ is precisely the first S rows in the transfer function $(I - Q)^{-1} P$.

(P2) *Hollowness*: As with Q , diagonal entries of Q_S must all be zero.

(P3) *Independence Pattern*: Barring exact cancellations⁵, $[Q_S]_{ij} \neq 0$, $i \neq j$ if and only if there exists a direct

⁴Since we can interpret edges as subsystems represented by SISO transfer functions, we can understand the net effect as the transfer function resulting from connecting all of the subsystems defined by the edges in the walk in sequence.

⁵Cancellations can be avoided by perturbing the non-zero links in (Q, P) by some small and random delta. If we allow exact cancellations, however, some edges that were previously required to be non-zero may become zero (though not the other way around). A more precise statement of the requirement to preserve structure when cancellations are allowed is that $[Q_S]_{ij}$ is the sum of the net effect of the direct edge connecting j to i with net effects of all walks from j to i passing through \bar{S} only, scaled to account for the removal of the diagonal links introduced in the immersion.

edge from j to i in $\Gamma(Q, P)$ or a walk from j to i that passes only through nodes in \bar{S} .

We write the immersion of some DSF (Q, P) as $(Q_S, P_S) = \mathcal{A}(Q, P | S)$, where S is the set of indices corresponding to outputs Y_S . \square

To define the immersion function $\mathcal{A}(Q, P | S)$, permute and rewrite (1) according to S to yield

$$\begin{bmatrix} Y_S \\ Y_{\bar{S}} \end{bmatrix} = \begin{bmatrix} Q_{SS} & Q_{S\bar{S}} \\ Q_{\bar{S}S} & Q_{\bar{S}\bar{S}} \end{bmatrix} \begin{bmatrix} Y_S \\ Y_{\bar{S}} \end{bmatrix} + \begin{bmatrix} P_{SS} \\ P_{\bar{S}\bar{S}} \end{bmatrix} U. \quad (7)$$

Solving for $Y_{\bar{S}}$, we get

$$Y_{\bar{S}} = (I - Q_{\bar{S}\bar{S}})^{-1} Q_{\bar{S}S} Y_S + (I - Q_{\bar{S}\bar{S}})^{-1} P_{\bar{S}\bar{S}} U. \quad (8)$$

Plugging this back into the first row of (7) gives

$$Y_S = W Y_S + V U, \quad (9a)$$

$$W = Q_{SS} + Q_{S\bar{S}}(I - Q_{\bar{S}\bar{S}})^{-1} Q_{\bar{S}S}, \quad (9b)$$

$$V = P_{SS} + Q_{S\bar{S}}(I - Q_{\bar{S}\bar{S}})^{-1} P_{\bar{S}\bar{S}}. \quad (9c)$$

At this point, (9a) is nearly in the form of the signal structure representation of the immersion. However, by definition of the signal structure, Q must be hollow, but W in general will not be hollow. To correct this, let

$$D_W \triangleq \text{diag}(W) = \text{diag}(Q_{S\bar{S}}(I - Q_{\bar{S}\bar{S}})^{-1} Q_{\bar{S}S}), \quad (10)$$

where the latter equality holds since Q_{SS} is hollow. Subtracting $D_W Y_S$ from both sides of (9a) gives

$$(I - D_W) Y_S = (W - D_W) Y_S + V U \implies \quad (11)$$

$$Y_S = (I - D_W)^{-1} (W - D_W) Y_S + (I - D_W)^{-1} V U \quad (12)$$

$$\triangleq Q_S Y_S + P_S U. \quad (13)$$

Since Q_S is square and hollow, we have the signal structure representation of the immersion $(Q_S, P_S) = \mathcal{A}(Q, P | S)$. We now show that the defined immersion function \mathcal{A} results in an immersion according to Definition 3.

Theorem 1 *Let (Q, P) be a proper well-posed DSF defining the relationship $Y = QY + PU$ and let S index some subset of the outputs of Y . Suppose that the sums of the net effect of all (infinite) walks in (Q, P) converge. Then $(Q_S, P_S) = \mathcal{A}(Q, P | S)$ is an immersion according to Definition 3. \square*

PROOF Dynamics are Preserved: Suppose that Q is partitioned according to (7). Then, we have that $G = (I - Q)^{-1} P$ is given by

$$\begin{bmatrix} Y_S \\ Y_{\bar{S}} \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} P_{SS} \\ P_{\bar{S}\bar{S}} \end{bmatrix} U,$$

where

$$N_{11} = [I - (Q_{SS} + Q_{S\bar{S}}(I - Q_{\bar{S}\bar{S}})^{-1} Q_{\bar{S}S})]^{-1} = (I - Q_S)^{-1},$$

$$N_{12} = -N_{11} Q_{S\bar{S}}(I - Q_{\bar{S}\bar{S}})^{-1} = (I - Q_S)^{-1} Q_{S\bar{S}}(I - Q_{\bar{S}\bar{S}})^{-1}.$$

Thus, the first $|S|$ rows of G are $N_{11} P_{SS} + N_{12} P_{\bar{S}\bar{S}}$. Then

$$\begin{aligned} Y_S &= [N_{11} P_{SS} + N_{12} P_{\bar{S}\bar{S}}] U \\ &= [(I - W)^{-1} P_{SS} \\ &\quad + (I - W)^{-1} Q_{S\bar{S}}(I - Q_{\bar{S}\bar{S}})^{-1} P_{\bar{S}\bar{S}}] U \\ &= (I - W)^{-1} [P_{SS} + Q_{S\bar{S}}(I - Q_{\bar{S}\bar{S}})^{-1} P_{\bar{S}\bar{S}}] U \\ &= (I - W)^{-1} V U, \end{aligned}$$

where \tilde{Q}_S and \tilde{P}_S are given in 9a. Notice also that

$$\begin{aligned} (I - Q_S)^{-1} P_S &= (I - (I - D_W)^{-1} (W - D_W))^{-1} (I - D_W)^{-1} V \\ &= ((I - D_W) - (W - D_W))^{-1} V \\ &= (I - W)^{-1} V, \end{aligned}$$

thus $Y_S = (I - W)^{-1} V U = (I - Q_S)^{-1} P_S$, as required.

Hollowness is Preserved: Q_S is hollow by construction.

The Independence Pattern is Preserved: Since we have assumed that $(I - Q)$ has an inverse (well-posedness) and that the sum of the net effects of all walks converge, we have, by the small gain theorem, that $(I - Q_{\bar{S}\bar{S}})^{-1}$ is the sum of all walks from any $\bar{i} \in \bar{S}$ to any $\bar{j} \in \bar{S}$. Thus, for $i, j \in S$, $W_{ij} = [Q_{SS} + Q_{S\bar{S}}(I - Q_{\bar{S}\bar{S}})^{-1} Q_{\bar{S}S}]_{ij}$ is the sum of the net effect from j to i with the net effects of all walks that pass only through \bar{S} . Hence, if no walks cancel, W_{ij} will be non-zero if and only if $\Gamma(Q, P)$ as a direct link from j to i or a walk from j to i passing only through nodes in \bar{S} . Furthermore, since D_W is diagonal, $[Q_S]_{ij} = [(I - D_W)^{-1} (W - D)]_{ij}$ will be non-zero if and only if W_{ij} is non-zero; thus the independence pattern is preserved. \blacksquare

Corollary 1 *Let (Q, P) be a proper DSF, and let $G = (I - Q)^{-1} P$ be the transfer function corresponding to this DSF. Suppose that $S = \{i\}$ indexes a single output of the DSF. Then $\mathcal{A}(Q, P | S) = (Q_S, P_S)$ where $Q_S = 0$ and $P_S = [G]_i$ with $[G]_i$ as the i 'th row of G . \square*

PROOF Follows immediately from Theorem 1 and the fact that, when $|S| = 1$, $Q_S = 0$, meaning that $[G]_i = (I - Q_S)^{-1} P_S = P_S$. \blacksquare

For completeness, we also wish to define a *realization* of a DSF.

Definition 4 With $(Q_S, P_S) = \mathcal{A}(Q, P | S)$ as the immersion of some DSF (Q, P) , we call (Q, P) a *realization* of (Q_S, P_S) . \square

Realizations of DSFs need not be unique; in particular there could potentially exist infinitely many unique $(\hat{Q}, \hat{P}) \neq (Q, P)$ such that $(Q_S, P_S) = \mathcal{A}(\hat{Q}, \hat{P} | S)$.

Recall that the identifiability index of a network (Q, P) —where Q is a $p \times p$ matrix of rational functions—is $p^2 - p$. Since the immersion is another network (Q_S, P_S) , where Q_S is a $|S| \times |S|$ matrix of rational functions, the identifiability index of this immersion is $|S|^2 - |S| \leq p^2 - p$, with equality only when $|S| = p$ (meaning our immersion is equal to our original network).

B. Stacked Immersions

We now wish to define a different type of abstraction—which we will call a *stacked immersion*—which reduces structural information without reducing the number of outputs modeled.

Definition 5 (The Stacked Immersion) Consider a dynamic network characterized by $Y = QY + PU$, and a partition of Y such that (possibly after re-ordering) $Y = [Y'_{S_1} \cdots Y'_{S_a}]'$ with corresponding set of index sets $T = \{S_1, \dots, S_a\}$. A *stacked immersion* of (Q, P) is the pair (Q_T, P_T) such that

$$Q_T = \begin{bmatrix} Q_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & Q_a \end{bmatrix}, \quad P_T = \begin{bmatrix} P_1 \\ \vdots \\ P_a \end{bmatrix}, \quad (14)$$

where $(Q_i, P_i) = \mathcal{A}(Q, P | S_i)$. \square

We write the stacked immersion of (Q, P) with respect to the partition T as $(Q_T, P_T) = \mathcal{F}(Q, P | T)$.

Note that if $Y = QY + PU$, by Theorem 1, we also have that $Y = Q_T Y + P_T U$, meaning that any stacked immersion of some DSF is dynamically equivalent to that DSF. Furthermore, we will have that Q_S is hollow since each Q_i is hollow. A stacked immersion of (Q, P) is thus a stacking of regular immersions across a full partitioning of the outputs so that we can come up with an alternate (Q_T, P_T) that preserves the same input-output mapping.

It is very important to note that the stacked immersion (Q_T, P_T) is syntactically equivalent to a DSF (Q, P) where $Q = Q_T$ and $P = P_T$. However, semantically, they are very different. As discussed in Section III-A, immersions preserve the independence pattern represented in a system; therefore, in a stacked immersion, the independence pattern is preserved within each diagonal block but hidden between blocks. In other words, for a DSF (and barring cancellations), a zero in an off-diagonal of Q entry means that the net effect of all walks between the corresponding outputs must be zero, whereas, in the stacked immersion, this constraint need not hold.

In short, the stacked immersion (Q_T, P_T) is *not* a DSF since the zeros in the off-diagonal blocks do not represent independence between manifest variables. Hence, we call (Q_T, P_T) a *multi-DSF*. It should also be noted that if Y is not partitioned, meaning that $T = \{\{1, \dots, p\}\}$, then $\mathcal{F}(Q, P | T) = (Q, P)$, meaning any DSF is a stacked immersion of itself, and therefore is a multi-DSF. Thus the set of DSFs is a subset of the set of multi-DSFs.

The identifiability index of our stacked immersion is equal to the sum of the identifiability indices of all immersions that have been stacked.

IV. A SPECTRUM OF MODELS

We now use the results of the previous sections to demonstrate the capability of stacked immersions to represent LTI systems with any desired level of structural information. To accomplish this, we will show that, for any arbitrary state

space model, there exists a DSF, called the full-state DSF, that is informationally equivalent to that model. We also show that this same DSF has a stacked immersion in one-to-one correspondence with the transfer function of that state space model. We begin with a definition.

Definition 6 (Full-State DSF) Suppose that we have a state space model (A, B, C, D) , where $C = I$ (i.e., we measure every state). With $A \triangleq [a_{ij}] \in \mathbb{R}^{p \times p}$ and $B \triangleq [b_{ij}] \in \mathbb{R}^{n \times m}$, and following the procedure contained in [21], we derive the DSF (Q, P) of this model as

$$Q = \begin{bmatrix} 0 & \frac{a_{12}}{s-a_{11}} & \cdots & \frac{a_{1p}}{s-a_{11}} \\ \frac{a_{21}}{s-a_{22}} & 0 & \cdots & \frac{a_{2p}}{s-a_{22}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_{p1}}{s-a_{pp}} & \frac{a_{p2}}{s-a_{pp}} & \cdots & 0 \end{bmatrix}, \quad (15)$$

$$P = \begin{bmatrix} \frac{b_{11}}{s-a_{11}} & \frac{b_{12}}{s-a_{11}} & \cdots & \frac{b_{1m}}{s-a_{11}} \\ \frac{b_{21}}{s-a_{22}} & \frac{b_{22}}{s-a_{22}} & \cdots & \frac{b_{2m}}{s-a_{22}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{b_{p1}}{s-a_{pp}} & \frac{b_{p2}}{s-a_{pp}} & \cdots & \frac{b_{pm}}{s-a_{pp}} \end{bmatrix} + (I - Q)D. \quad (16)$$

We call (Q, P) in equations (15) and (16) the *full-state DSF*. \square

Theorem 2 Let (A, B, C, D) with $C = I$, and let (Q, P) be the (full-state) DSF corresponding to this state space model. Then (Q, P) is the unique immersion of (A, B, C, D) and (A, B, C, D) is the unique minimal realization of (Q, P) . \square

PROOF First, (Q, P) can be recovered uniquely from (A, B, C, D) regardless of the choice of C (see [21]). To recover (A, B, C, D) from (Q, P) , first note that every entry in Q is a strictly proper transfer function with a constant in the numerator and one pole in the denominator. The pole is consistent in each row, and is the value for the diagonal entry of that row. The numerator on the off-diagonals is the value of the off-diagonals in A , and so we can recover A uniquely from Q .

Since Q is strictly proper, $(I - Q)$ will have a proper inverse [6], [7]. Let $P = \tilde{P} + (I - Q)D$. Then $(I - Q)^{-1}P = (I - Q)^{-1}\tilde{P} + D$. Since \tilde{P} is strictly proper from (16), we have that $(I - Q)^{-1}\tilde{P}$ will be strictly proper. Furthermore, since it is strictly proper, we have that $\lim_{s \rightarrow \infty} (I - Q)^{-1}\tilde{P} = 0$. Thus $\lim_{s \rightarrow \infty} (I - Q)^{-1}P = D$ and we have recovered D uniquely. Subtract $(I - Q)D$, which is known, from P to get \tilde{P} . The numerators of \tilde{P} specify B uniquely.

Finally, no state space realization of a smaller order can generate (Q, P) of the given dimensions; thus the (A, B, C, D) found must be minimal. \blacksquare

Thus, by Theorem 2, a full-state DSF is informationally equivalent to a (minimal) state space representation.

Remark 1 Definition 6 and Theorem 2 both deal with the case where $C = I$. In the more general case (A, B, C, D) where C is square, full-rank, and known, we can perform a change of basis with $T = C$ to a new system $(T^{-1}AT, TB, CT^{-1}, D) \triangleq (\hat{A}, \hat{B}, I, D)$. Thus, if we know

C (which is necessary to reconstruct a state space model from a DSF), we can assume that $C = I$ without loss of generality. \square

We now look at the other end of the spectrum, the transfer function. Again, we begin with a definition.

Definition 7 (Final Immersion) A *final immersion* of a proper DSF (Q, P) is the stacked immersion $F_T = \mathcal{F}(Q, P | T)$ where $T = \{\{1\}, \dots, \{p\}\}$. \square

Theorem 3 Let (Q, P) be a proper DSF with transfer function $G = (I - Q)^{-1}P$, and suppose that $T = \{\{1\}, \dots, \{p\}\}$. Then the final immersion $\mathcal{F}(Q, P | T) = (0, G)$. \square

PROOF By Corollary 1, we have that $(Q_F, P_F) = \mathcal{F}(Q, P | \{\{1\}, \dots, \{p\}\})$ is given by

$$Q_F = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} = 0, \quad P_F = \begin{bmatrix} G_1 \\ \vdots \\ G_p \end{bmatrix} = G, \quad (17)$$

where G_i is the i 'th row of G . \blacksquare

Theorem 3 shows us that the final immersion is in one-to-one correspondence with the transfer function of the full-state DSF. Furthermore, the transfer function of the full-state DSF is also the same as the transfer function of its corresponding state-space realization.

Thus Theorem 2 and Theorem 3 shows us that there exists a multi-DSF containing as much structural information as the state space, and a multi-DSF containing as little as the transfer function. By choosing other partitions of the outputs Y and taking a stacked immersion with that partition, we can create other multi-DSFs containing intermediate levels of information. As a result, the multi-DSF is a general model of LTI systems, capable of representing many levels of structural knowledge, many more than state space models or transfer functions alone are capable of doing. These different abstractions form a spectrum of models of LTI systems containing various levels of structural information.

Since the full-state DSF has Q as an $n \times n$ matrix of rational functions, the identifiability index for a full-state DSF is $n^2 - n$. Since the final immersion is the stacking of n immersions, each with $|S| = 1$, the identifiability index of the final immersion is $\sum_{i=1}^n (1^2 - 1) = 0$. Since the identifiability index is always non-negative, it is minimized at the final immersion. Since the identifiability index always decreases for non-trivial immersions, every non-trivial stacked immersion of the full-state DSF has an identifiability index strictly between 0 and $n^2 - n$; thus the identifiability index forms a partial ordering over multi-DSFs placing them on the spectrum of structural informativity ranging from the state-space model to the transfer function.

V. ACKNOWLEDGEMENTS

We would like to thank Paul Van den Hof and Arne Dankers for their deep insights into this subject.

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